ACADEMIC
TASK FORCE

MATHEMATICS SPECIALIST

Year 11 ATAR COURSE

Units 1 and 2

WACE Revision Series

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The Mathematics Specialist Units 1 & 2 Revision Series provides a comprehensive set of revision/review questions for the West Australian Mathematics Specialist Units 1 & 2. It is accompanied by a set of fully worked solutions, which doubles as a set of model answers.

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Mathematics Specialist Revision Series Units 1 & 2

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Fully Worked Solutions

Mathematics Specialist Revision Series
Units 1 & 2

- The Mathematics Specialist Revision Series Units 1 & 2 provides a comprehensive set of revision/review questions for the new year 11 Mathematics Specialist Units 1 & 2 course.

- The review questions are written at test/examination level for both the Calculator Free and Calculator Assumed Sections and presented in a write-on format in topical order.

- This book exposes students to questions and problems at test/examination level.

- These questions are suitable for end-of-topic reviews and pre-test and pre-examination reviews.

- It is accompanied by a set of fully worked solutions with which students can measure their solutions. These solutions are often not the only solutions but they provide a model for students to work with. Students, interrogate your solutions to understand your errors and your successes. It may sometimes to possible to achieve a correct numerical answer with faulty reasoning!

- Do not memorise solutions. Understand the techniques and processes used in relation to the questions asked.

Notes

Combinatorics

- $^nC_r \equiv \binom{n}{r} \equiv \dfrac{n!}{r!\,(n-r)!}$

  $= \dfrac{\overbrace{n\times(n-1)\times(n-2)\times......\times(n-r+1)}^{r\ \text{terms}}}{r!}$

- $^nC_r = {^nC_{n-r}}$
- $^nC_r + {^nC_{r+1}} = {^{n+1}C_{r+1}}$
- $^nC_0 = {^nC_n} = 1$
- $^nC_1 = {^nC_{n-1}} = n.$
- $^nC_0 + {^nC_1} + {^nC_2} + \ldots + {^nC_n} = 2^n$.
- $^nP_r = \dfrac{n!}{(n-r)!} = \underbrace{n\times(n-1)\times(n-2)\times.....(n-r+1)}_{r\ \text{terms}}$
- If Tasks A and B are mutually exclusive, then:
  - A or B can be completed in n(A) + n(B) ways.
  - A & B can be completed in n(A) × n(B) ways.
- r objects can be chosen from n unlike objects:
  - without replacement in ^nC_r ways.
  - with replacement in n^r ways.
- r items from n items of type I and s items from m items of type II, can be chosen and then arranged in $^nC_r \times {^mC_s} \times (r+s)!$ ways.
- n unlike items can be arranged in a line in $n!$ ways.
- r objects out of n objects all different $(r \le n)$ may be arranged in a line, with no object used more than once in ^nP_r or $^nC_r \times r!$ ways.
- n unlike objects can be arranged in a line with 2 specific objects apart from each other in $n! - (n-1)! \times 2!$ ways.
- n unlike items can be arranged in a line with r specific items adjacent to each other in $(n-r+1)! \times r!$.

Inclusion-Exclusion Principle

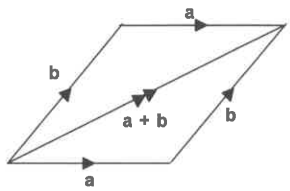
- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C)$
  $- n(A \cap B) - n(A \cap C) - n(B \cap C)$
  $+ n(A \cap B \cap C)$

Pigeon Hole Principle

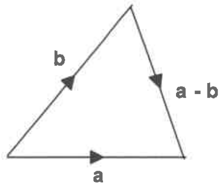
- If m items are placed in n containers, where $m > n$, at least one container will contain:
  - more than one item.
  - no more than **Int**$\left(\dfrac{m}{n}\right)$ items.
  - at least **Int**$\left(\dfrac{m}{n}\right) + 1$ items.

Vectors

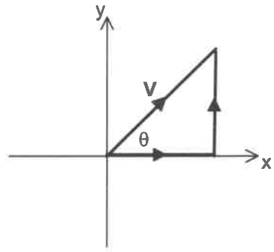
- Addition of vectors



Subtraction of vectors



Vectors (resolving into components)



Component along

x – axis $v_x = v\cos\theta$

y – axis $v_y = v\sin\theta$

$v = v\cos\theta\,\mathbf{i} + v\sin\theta\,\mathbf{j}$

$= \langle v\cos\theta,\ v\sin\theta\rangle$

$= \begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$

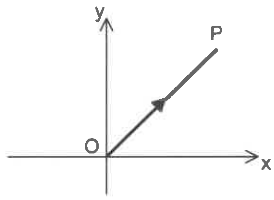
Vectors (magnitude and direction)

$\boldsymbol{u} = x\,\mathbf{i} + y\,\mathbf{j}$

- Magnitude of $\boldsymbol{u}$ $|\boldsymbol{u}| = \sqrt{(x^2 + y^2)}$
- Direction of $\boldsymbol{u}$ $= \tan^{-1}\dfrac{y}{x}$ with positive x-axis

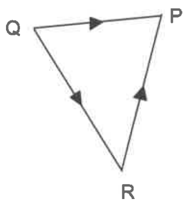
  (*Locate quadrant location first*)
- Unit vector in the direction of $\boldsymbol{u}$ is $\hat{\boldsymbol{u}} = \dfrac{1}{|\boldsymbol{u}|}\,\boldsymbol{u}$

Position Vectors



P is the point (x, y).

- The position vector of P with respect to the origin O is **OP** or $\overrightarrow{OP}$.
- **PQ = OQ – OP**



P, Q and R are points on the x-y plane (in space).

- **QP = QR + RP**

Parallel vectors

- If $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel $\Leftrightarrow$ $\boldsymbol{u} = \lambda\,\boldsymbol{v}$.
- For $\boldsymbol{u} = \lambda\,\boldsymbol{v}$, if $\lambda > 0$ then $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel and in the same direction.
- For $\boldsymbol{u} = \lambda\,\boldsymbol{v}$, if $\lambda < 0$ then $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel and in the opposite direction.

Perpendicular vectors

- If $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular $\Leftrightarrow$ then $\boldsymbol{u}\,.\,\boldsymbol{v} = 0$.

Relative Vectors

- The position vector or displacement vector of P relative to Q is given by:

  $_P r_Q = \mathbf{OP} - \mathbf{OQ} = r_P - r_Q$
- The velocity vector of P relative to Q is given by: $_P v_Q = v_P - v_Q$

Scalar Product

Let $u = a\,\mathbf{i} + b\,\mathbf{j}$ and $v = x\,\mathbf{i} + y\,\mathbf{j}$

- scalar product $u \bullet v = ax + by$
- Angle between u and v: $\cos\theta = \dfrac{u \cdot v}{|u||v|}$

Projections

- The scalar projection of u onto v

  $= |u|\cos\theta = |u| \times \dfrac{u \bullet v}{|u||v|} = u \bullet \hat{v}$.
- The vector projection of u onto v,

  $\text{proj}_v\, u = (|u|\cos\theta)\,\hat{v} = (u \bullet \hat{v})\,\hat{v}$.
- The vector rejection of u onto v,

  $\text{rej}_v\, u = u - \text{proj}_v\, u = u - (u \bullet \hat{v})\,\hat{v}$

Work Done

- The work done by a force of F Newtons through a distance of s metres is given by: Scalar Projection of F along direction of motion $\times\, s$ Joules.
- The work done by a force F Newtons in moving a body over a displacement d metres is given by: Work done = $F \bullet d$ Joules.

Collision (interception) and closest approach

Displacement Method

- Position vector of A and B at time t:

  $r_A = a + t\,u \quad r_B = b + t\,v$
- For collision (interception) at time t: $r_A = r_B$.

  Compare x components and solve for t, t_x.

  Compare y components and solve for t, t_y
- If $t_x = t_y$, $\Rightarrow$ collision (interception) occurs at t_x.

  If $t_x \neq t_y$ $\Rightarrow$ there is no collision (interception).
- Displacement vector between A and B at time t

  $d = r_A - r_B$
- Distance between A and B at time t, $s = |d|$.
- If $s = 0$ has a real solution for t, then there is collision (interception).

Relative Vectors Method

- For collision (interception) at time t:

  $_B r_A = t\;_A v_B$ where $t > 0$

  i.e. $b - a = t\,(u - v)$

  Compare components and solve for t_x and t_y.
- If $t_x = t_y$, $\Rightarrow$ collision (interception) occurs at t_x.

  If $t_x \neq t_y$ $\Rightarrow$ there is no collision (interception).

Scalar product method

- At closest approach, $_B r_A$ is perpendicular to $_B v_A$

  $_B r_A \bullet {}_B v_A = 0$

  Solve the resulting equation for t.

  This is the time of closest approach.
- Substitute value of t into $|_A r_B|$ to obtain the distance of the closest approach.
- If $|_A r_B| = 0$ on substituting value of t, then there is collision (interception).

Exact Values

θ°	θ rad	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	$\to\infty$

General Solutions

Equation	General Solution $n \in \mathbb{Z}$
$\sin x = k$	$x = (-1)^n \sin^{-1} k + n\pi$ *OR* $x = \sin^{-1} k + 2n\pi$, or $-\sin^{-1} k + (2n+1)\pi$
$\cos x = k$	$x = 2n\pi \pm \cos^{-1} k$
$\tan x = k$	$x = \tan^{-1} k + n\pi$

Trigonometric Identities

- secant A $= \dfrac{1}{\cos A}$
- cosecant A $= \dfrac{1}{\sin A}$
- cotangent A $= \dfrac{1}{\tan A}$
- $\sin^2\theta + \cos^2\theta = 1$
- $\sec^2\theta = 1 + \tan^2\theta$
- $\text{cosec}^2\theta = 1 + \cot^2\theta$
- $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$
- $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
- $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

Trigonometric Identities

- $\sin 2A = 2 \sin A \cos A$
- $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$
  $= 1 - 2\sin^2 A$
- $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$
- $\sin 3A = 3 \sin A - 4 \sin^3 A$
- $\cos 3A = 4\cos^3 A - 3 \cos A$
- $\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$
- $\sin A \sin B = -\frac{1}{2}[\cos(A + B) - \cos(A - B)]$
- $\sin A \pm \sin B = 2 \sin\left(\dfrac{A \pm B}{2}\right) \cos\left(\dfrac{A \mp B}{2}\right)$
- $\cos A + \cos B = 2 \cos\left(\dfrac{A + B}{2}\right) \cos\left(\dfrac{A - B}{2}\right)$
- $\cos A - \cos B = -2 \sin\left(\dfrac{A + B}{2}\right) \sin\left(\dfrac{A - B}{2}\right)$
- $a \sin x \pm b \cos x \equiv \sqrt{a^2+b^2}\ \sin\left[x \pm \tan^{-1}\left(\dfrac{b}{a}\right)\right]$
- $a \cos x \pm b \sin x \equiv \sqrt{a^2+b^2}\ \cos\left[x \mp \tan^{-1}\left(\dfrac{b}{a}\right)\right]$

Trigonometric Graphs

	$y = a \sin(bx + c) + d$ $y = a \cos(bx + c) + d$
Mean Line	$y = d$
Amplitude	$\lvert a \rvert$
Min./Max. y	Min: $d - \lvert a \rvert$, Max: $d + \lvert a \rvert$
Period	$360^\circ/b$ or $2\pi/b$
Phase shift	*Shifted c/b degrees/radians to the left*

	$y = a \tan(bx + c) + d$
Mean Line	$y = d$
Period	$180^\circ/b$ or π/b
Phase shift	*Shifted c/b degrees/radians to the left*

Matrices

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$
- In general, **AB** ≠ **BA**.
- If **AB** = **BA** = **I**, then:
  - $\mathbf{A}^{-1} = \mathbf{B}$
  - $\mathbf{B}^{-1} = \mathbf{A}$
- $\mathbf{AB} = \mathbf{BA} = k\mathbf{I} \Rightarrow \mathbf{A}^{-1} = \frac{1}{k}\mathbf{B}$ and $\mathbf{B}^{-1} = \frac{1}{k}\mathbf{A}$.

- For $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:
  - $\det(\mathbf{A}) = \lvert \mathbf{A} \rvert = ad - bc$
  - $\mathbf{A}^{-1}$, exists only if $\lvert \mathbf{A} \rvert \neq 0$ and is given by $\mathbf{A}^{-1} = \dfrac{1}{\lvert \mathbf{A} \rvert}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
- If $\mathbf{A}^{-1}$ does not exist then **A** is **singular** or **non-invertible** [In which case $\lvert \mathbf{A} \rvert = 0$].
- $\mathbf{AX} = \mathbf{B} \Rightarrow \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ provided $\mathbf{A}^{-1}$ exists.

Linear Transformations

- A linear transformation maps the point (0, 0) to the point (0, 0) and a set of parallel lines to another set (not necessarily the same) of parallel lines.
- Some transformation matrices:

Transformation	Matrix
Reflection about the x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection about the y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection about the line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Reflection about the line $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
Reflection about $y = x \tan\theta$, $\theta \neq (2n+1)\pi/2$	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
Rotation 90° clockwise about the origin	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Rotation 90° anti-clockwise about the origin	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Rotation 180° clockwise about the origin	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
Rotation θ° anti-clockwise about the origin	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
Dilation factor $k > 0$ along the x-axis	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
Dilation factor $k > 0$ along the y-axis	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
Enlargement factor $k > 0$ about the origin	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

- The absolute value of the determinant of a transformation matrix is defined as the scale factor for area, which is the ratio $\frac{\text{Area of Image}}{\text{Area of Object}}$.

Complex numbers

- $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$.
- If $x + y\,i = a + b\,i$, then $x = a$, and $y = b$.
- $(x + y\,i) \pm (a + b\,i) = (x \pm a) + (y \pm b)\,i$
- $(x + y\,i)(a + b\,i) = (xa - yb) + (xb + ya)\,i$
- The **complex conjugate** of $z = x + y\,i$ is $\overline{z} = x - y\,i$. Note that, $z\overline{z} = x^2 + y^2$
- $\overline{u \pm v} = \overline{u} \pm \overline{v}$ and $\overline{u \times v} = \overline{u} \times \overline{v}$
- $\frac{1}{a+bi} = \frac{1}{a+bi} \times \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}$
- $\frac{x+yi}{a+bi} = \frac{x+yi}{a+bi} \times \frac{a-bi}{a-bi} = \frac{(x+yi)(a-bi)}{a^2+b^2}$

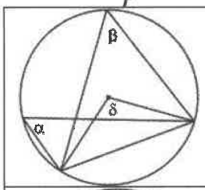
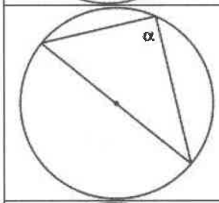
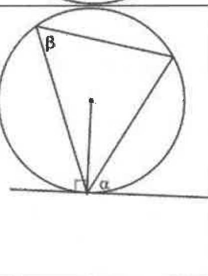
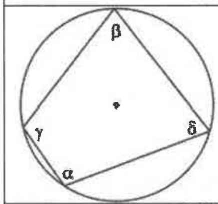
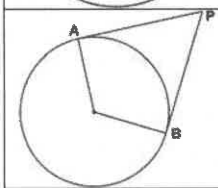
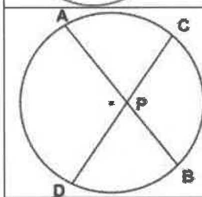
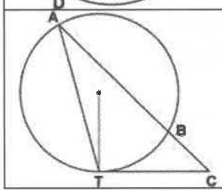
Similar Triangles

- AA Test
  Two angles of the first triangle are equal to two angles of the second triangle.
- SSS Test
  Ratios of corresponding side lengths are equal.
- SAS Test
  Ratios of the two corresponding side lengths are equal and included angles are equal.
- RHS Test
  Right Triangles with ratios of hypotenuse and one other corresponding side equal.

Congruent Triangles

- SSS Test
  All three corresponding sides of the two triangles have the same side length.
- SAS Test
  Two sides and the included angle of both triangles are the same.
- ASA Test
  Two angles and the included side of both triangles are equal.
- AAS Test
  Two angles and a corresponding non-included side of both triangles are equal.
- RHS Test
  Right Triangles with hypotenuse and one corresponding side equal.

Circle Properties

	$\alpha = \beta$ Angles in the same segment $\delta = 2\beta$ Angle at centre is twice angle at the circumference
	$\alpha = 90°$ Angle in a semi-circle = 90°.
	Angle between radius and tangent = 90°. $\alpha = \beta$ Angle between a tangent and a chord is equal to the angle in the alternate segment.
	$\alpha + \beta = 180°$ $\gamma + \delta = 180°$ Opposite angles in a cyclic quadrilateral are supplementary.
	PA = PB Length of tangents to a circle from a common external point are the same.
	AP × PB = CP × PD
	AC × BC = TC^2

01 Combinatorics I

Calculator Free

1. [9 marks: 2, 2, 2, 3]

   Find k if:

   (a) $\frac{10!}{k!} = 720$

   (b) $\frac{k!}{8!} = 990$

   (c) ${}^{k}P_3 = 210$

   (d) $\frac{10! - k!}{k!} = 30\,239$

Calculator Free

2. [4 marks: 1, 1, 1, 1]

The letters of the word RICHES are rearranged in a line. No letter is used more than once. Write mathematical expressions for:

(a) the total number of possible arrangements.

(b) the number of arrangements with the letters I and E are adjacent.

(c) the number of arrangements with the letters I and E not adjacent

(d) the number of arrangements where the vowels are adjacent and the letters C and H are adjacent.

---

3. [4 marks: 1, 1, 1, 1]

Codes are formed by using the letters of the word MATHS, with no letter being used more than once:

(a) How many five letter codes start with M?

(b) How many five letter codes have the letters A and M together?

(c) How many four letter codes start with the letter M ?

(d) How many three letter codes end with the letter S.

Calculator Assumed

4. [4 marks: 1, 1, 1, 1]

Three year 11 students and four year 12 students are to be arranged in a line.

(a) In how many ways can this be done?

(b) In how many arrangements are all the year 11 students next to each other?

(c) How many arrangements have all students of the same year group adjacent to each other?

(d) How many arrangements have no student of the same year group adjacent to each other?

---

5. [9 marks: 1, 2, 3, 3]

Six digit numbers are formed using the digits 1, 2, 3, 4, 5, and 6. Each digit is used only once.

(a) How many six digit numbers are possible?

(b) How many six digit even numbers are possible?

(c) How many five digit even numbers greater than 40 000 are possible?

(d) How many even numbers greater than 40 000 are possible?

Calculator Assumed

6. [7 marks: 2, 2, 3]

Passwords consisting of between 8 and 12 characters inclusive are to be created using the letters of the alphabet (case sensitive) and the digits 0 to 9 inclusive.

(a) Write mathematical expressions for the number of possible passwords if:

(i) no character can be used more than once.

(ii) repetition of characters are permitted.

(b) A computer program is capable of checking 4 billion (1×10^9) passwords per second. How long will the computer take to check all the possible passwords in part (a) (ii)? Give your answer in years.

---

7. [4 marks: 2, 2]

The length of a password is determined by the number of characters in the password. Using the digits 0 to 9 inclusive and the case sensitive letters of the alphabet, find the minimum length of a password if the number of possible passwords is to exceed 1 trillion (1×10^{12}) if:

(a) repetition of characters is not permitted.

(b) repetition of characters is permitted.

02 Combinatorics II

Calculator Assumed

1. [5 marks: 1, 2, 2]

A random survey involving 200 students revealed the following. 38 students did not have any calculator (scientific or CAS) with them. 142 students had a CAS calculator with them. 52 students had a scientific calculator with them. How many students had:

(a) at least one calculator with them?

(b) both types of calculators with them?

(c) exactly one type of calculator with them?

---

2. [5 marks: 3, 2]

80 students were asked in a survey if they had previously broken a leg and/or broken an arm. Information from the survey indicated that 40% of students had suffered broken arms, one quarter of those surveyed had suffered broken legs and 40% had not previously had a broken arm or leg. How many students had:

(a) broken an arm and a leg?

(b) broken an arm but not a leg?

Calculator Assumed

3. [4 marks: 1, 1, 2]

In a group of 40 students, there were 10 boys who were colour vision deficient (CVD) and 15 girls who were not CVD. There were as many boys who were not CVD as there were boys who were CVD. How many of these students:

(a) were boys?

(b) were either boys or colour vision deficient?

(c) were colour vision deficient?

---

4. [5 marks: 2, 2, 1]

In a survey of teachers teaching mathematics to year nines, 15 teachers had a mathematics degree, 55 men teachers did not have a mathematics degree and 10 women teachers had a mathematics degree. There were as many teachers who were either female or had a mathematics degree as there were men teachers.

(a) How many male teachers were there?

(b) How many female teachers were there?

(c) How many teachers were surveyed?

Calculator Assumed

5. [7 marks: 1, 1, 1, 1, 2, 1]

Flags of 10 different nations including that of Australia and New Zealand are to be flown from 10 flag poles set in a line. The flag poles are labelled poles 1 to 10. How many ways are there of assigning a flag to each of these poles if the:

(a) Australian flag must be flown from pole 1?

(b) Australian flag or New Zealand flag must be flown from pole 1?

(c) Australian flag and the New Zealand flag must be flown from poles 1 and 10 respectively?

(d) Australian flag must be flown from pole 1 and the New Zealand flag must not to be flown from pole 10.

(e) Australian flag must be flown from pole 1 or the New Zealand flag must be flown from pole 10.

(f) Australian flag must be flown from pole 1 or the New Zealand flag must be flown from pole 10 but not both at the same time.

Calculator Assumed

6. [6 marks: 2, 2, 2]

Twelve different coloured light bulbs, including a red bulb and a blue bulb are to be fitted into bulb sockets installed along a straight edge of a patio running East to West. The bulb socket at the extreme Eastern end is labelled E and the bulb socket at the extreme Western end is labelled W. Determine the number of arrangements with:

(a) the red light bulb not fitted into bulb sockets E or W.

(b) the red light bulb and the blue light bulb not fitted into bulb sockets E or W.

(c) the red light bulb or the blue light bulb not fitted into bulb sockets E or W.

7. [7 marks: 2, 2, 3]

Ten potted plants including four pots of roses of different shades of red and three pots of azaleas (each of a different colour) are to be arranged in a line along a footpath. How many arrangements will have:

(a) the potted roses adjacent to each other?

(b) the potted azaleas adjacent to each other?

(c) the roses adjacent to each other or the azaleas adjacent to each other?

Calculator Assumed

8. [6 marks: 2, 2, 2]

An analysis was conducted on the tickets bought by 150 patrons of the Perth International Festival. The survey was restricted to the three categories: Classical Music, Contemporary Music, Theatre, Circus & Dance (TCD).

- All had bought tickets to at least one of these categories.
- 30 had tickets to all categories.
- A total of 100 had tickets to a Classical Music event.
- A total of 90 had tickets to a Contemporary Music event.
- 80 had tickets to a Classical Music event as well as a TCD event.
- 50 had tickets to a Classical Music as well as a Contemporary Music event.
- 30 had tickets to a Contemporary Music as well as a TCD event.

(a) How many patrons had tickets to a Classical Music and a Contemporary Music event but not a Theatre, Circus & Dance event?

(b) How many patrons had tickets to only a Classical Music event?

(c) How many patrons had tickets to a Theatre, Circus & Dance event?

Calculator Assumed

9. [6 marks: 3, 3]

A survey was conducted on 100 travellers at an airport.

- 62 had been vaccinated against yellow fever (Y).
- 80 had been vaccinated against typhoid (T).
- 82 had been vaccinated against malaria (M).
- 8 had not been vaccinated against any of these three diseases.
- All who had vaccinations had been vaccinated against more than one of these three diseases.

(a) How many travellers were vaccinated against yellow fever and typhoid but not malaria?

(b) How many travellers were vaccinated against yellow fever, typhoid and malaria?

Calculator Assumed

10. [6 marks: 1, 4, 1]

50 students were asked if they had attended the open-day sessions at the University of Western Australia (W), Murdoch University (M) and Curtin University (C).

- 5 students did not attend any of these sessions.
- 20 students attended the open day session at W.
- 23 students attended the open day session at M.
- 30 students attended the open day session at C.
- 3 students attended the open day sessions at W and M but not at C.
- 7 students attended the open day sessions at C and M but not at W.
- 12 students attended the open day sessions at C and W but not at M.

(a) How many students attended sessions at exactly 2 of these universities?

(b) How many students attended sessions at no more than 2 of these universities?

(c) How many students attended sessions at exactly one of these 3 universities?

Calculator Assumed

11. [8 marks: 1, 1, 1, 2, 1, 2]

Seven students including Amy, Brian and Catherine are to be arranged in a line. How many possible arrangements are there with:

(a) Amy or Brian or Catherine on the extreme left?

(b) Amy on the extreme left?

(c) Amy on the extreme left and Catherine on the extreme right?

(d) Amy on the extreme left or Catherine on the extreme right?

(e) Amy on the extreme left, Brian in the middle and Catherine on the extreme right?

(f) Amy on the extreme left or Brian in the middle or Catherine on the extreme right?

Calculator Assumed

12. [11 marks: 1, 1, 2, 3, 2, 2]

Ten different coloured balls, including a red, a blue and a green ball, are to be placed in ten different boxes labelled A to J. One ball is to be placed in each box. How many ways are there of placing the balls (one in each box) with:

(a) the red ball in box A?

(b) the red ball in box A and the blue ball in box B?

(c) the red ball in box A or the blue ball in box B?

(d) the red ball in box A or the blue ball in box B or the green ball in box C?

(e) the red ball in box A but the blue ball not in box B?

(f) the green ball in box C but the red ball not in box A and the blue ball not in box B?

Calculator Assumed

13. [8 marks: 1, 1, 1, 1, 2, 2]

Consider the set of integers between 1 000 and 9 999 inclusive.
How many integers in this set:

(a) are divisible by 2?

(b) are divisible by 3?

(c) are divisible by 2 and 3?

(d) are divisible by 2 and 6?

(e) are divisible by 2 or 3?

(f) are divisible by 2 or 3 but not both?

Calculator Assumed

14. [7 marks: 1, 1, 1, 2, 2]

Consider the set of integers between 500 and 5 000 inclusive.
How many integers in this set:

(a) are divisible by 5?

(b) are divisible by 10?

(c) are divisible by 5 and 10?

(d) are divisible by 5 or 10?

(e) are divisible by 5 or 10 and is an even number?

Calculator Assumed

15. [6 marks: 3, 3]

Consider the set of integers between 1 000 and 5 000 inclusive.

(a) Complete the following table listing the number of multiples of n within this set for the given values of n.

n	Number of multiples of n in this set.
2	
3	
5	
6	
10	
15	
30	

(b) Find the number of integers in this set that are multiples of 2, 3 or 5.

Calculator Assumed

16. [6 marks: 3, 3]

Consider the set of integers between 500 and 5 000 inclusive.

(a) Complete the following table.

Multiples of	Number of multiples in this set.
3	
5	
7	
3 and 5	
3 and 7	
5 and 7	
3 and 5 and 7	

(b) Find the number of integers in this set that are multiples of 3, 5 or 7.

Calculator Assumed

17. [6 marks: 3, 3]

Consider the set of integers between 2500 and 10 000 inclusive.

(a) Complete the following table.

Multiples of	Number of multiples in this set.
2	
4	
5	
2 and 4	
2 and 5	
4 and 5	
2 and 4 and 5	

(b) Find the number of integers in this set that are multiples of 2, 4 or 5.

03 Combinatorics III

Calculator Free

1. [8 marks: 2, 2, 2, 2]

(a) Determine the $\binom{9}{6}$.

(b) Evaluate $\binom{9}{3} + \binom{9}{4}$.

(c) Simplify ${}^{10}C_5 \times {}^5P_5$. You are not required to evaluate your answer.

(d) Given that ${}^5P_{r+1} = {}^5P_r$, find r.

Calculator Free

2. [13 marks: 2, 2, 2, 4, 3]

(a) Find n and r if ${}^{n}C_{r} = \dfrac{n \times (n-1) \times 98}{3 \times 2 \times 1}$.

(b) Find n and r if ${}^{n}P_{r} = 20 \times 19 \times 18 \times 17$

(c) Find a and b if ${}^{30}C_{a} = {}^{3a}C_{b}$.

(d) Find all possible values of a and b if ${}^{12}C_{a} = {}^{12}C_{2a+b}$.

(e) Find a possible set of values for a and b if $10 \times {}^{9}P_{4} = 6 \times {}^{a}P_{b}$.

Calculator Free

3. [3 marks]

(a) How many ways are there of choosing 2 students from a group of 100 students?

(b) How many ways are there of choosing 98 students from a group of 100 students?

4. [7 marks: 2, 2, 3]

A media folder has 10 video-clips. How many ways are there of choosing:

(a) seven of these clips?

(b) seven or eight of these clips?

(c) at least one of these clips?

Calculator Assumed

5. [12 marks: 1, 3, 2, 3, 3]

A committee of 9 people are to be selected from 10 Labor, 8 Liberal and 5 Green politicians. How many different ways can the committee be selected if:

(a) there are no restrictions

(b) all three political parties are equally represented

(c) there are no Green representatives

(d) the Liberal representatives are in the (absolute) majority

(e) a husband and wife pair, Alex and Alice, cannot be in the same committee.

Calculator Assumed

6. [10 marks: 1, 2, 3, 4] [TISC]

Wei has a collection of 20 stickers in her pink box and 25 stickers in her blue box. All these stickers are different from each other.

(a) In how many ways can Wei pick 3 stickers from her pink box?

(b) In how many ways can Wei pick 2 stickers from her blue box and arrange them in a line?

(c) In how many ways can Wei pick 3 stickers from her pink box and 2 stickers from her blue box and arrange them in a line if:

(i) there are no restrictions as to how the stickers are arranged?

(ii) all the stickers from the blue box must be together?

Calculator Free

7. [12 marks: 1, 2, 2, 2, 2, 3]

In how many ways can the letters of the word **COMBINE** be arranged in a straight line (no letter may be used more than once):

(a) using all seven letters?

(b) using all seven letters and starting with the letter **C**?

(c) using all seven letters and ending in **BONE?**

(d) using only 5 letters at a time?

(e) using all seven letters with the vowels in the first three places (from the left)?

(f) using only two vowels and two consonants?

Calculator Free

8. [12 marks: 1, 2, 2, 3, 4]

Consider the letters of the word CONQUEST. Write mathematical expressions for the number of ways the letters of this word can be *rearranged*:

(a) if the first letter must be a consonant.

(b) if the first three letters must be consonants.

(c) if one of the first three letters must be a consonant.

(d) if at least one of the first three letters must be a consonant.

(e) if the vowels are to be sandwiched between two consonants (that is, a vowel must be preceded by a consonant and this vowel must also be followed by a consonant).

Calculator Free

9. [5 marks: 1, 1, 3] [TISC]

Jane wishes to create a password using the digits 1, 2, 3, 4, 5, 6 and the letters of her name J, A, N and E.

(a) How many eight character passwords can be created, if no character is used more than once?

(b) How many of the passwords in (a) include the letters J, A, N and E?

(c) How many passwords in (a) have digits adding up to exactly ten.

10. [6 marks: 3, 3]

A number is divisible by four if the last two digits of the number is divisible by four. For example, 4564 is divisible by four because the last two digits "64" is divisible by four; but 4502 is not divisible by four because the last two digits "02" is not divisible by four. Using the digits 0 to 9 inclusive, write an expression for the number of five digit numbers divisible by four that can be formed:

(a) if digits may be repeated?

(b) if no digit is to be repeated?

Calculator Assumed

11. [8 marks: 1, 1, 2, 2, 2]

A contingent of six athletes is to formed from 10 swimmers, 12 cyclists and 8 gymnasts. How many different contingents can be formed if each contingent must have:

(a) 2 gymnasts?

(b) 3 swimmers

(c) 2 gymnasts and 3 swimmers

(d) 2 gymnasts or 3 swimmers?

(e) 2 gymnasts or 3 swimmers but not both?

Calculator Assumed

12. [7 marks: 3, 2, 2]

Five books are to be selected and arranged on a library display shelf. These five books are to be selected from a collection of 10 adult novels, 5 non-fiction books and 8 illustrated children's books.

(a) Complete the table below.

Composition of books on display shelf	Number of different arrangements
2 adult novels	
3 illustrated children's books	
2 adult novels and 3 illustrated children's book	

(b) Determine the number of possible arrangements with either 2 adult novels or 3 illustrated children's books.

(c) Determine the number of possible arrangements with either 2 adult novels or 3 illustrated children's books but not both.

Calculator Assumed

13. [8 marks: 3, 2, 3]

A password consisting of four characters is to be chosen from 10 digits, 26 upper case letters, 26 lower case letters and a set of 32 symbols.

(a) Complete the table below.

Composition of password	Number of different passwords
2 lower case letters	
2 symbols	
2 lower case letters and 2 symbols	

(b) Determine the number of possible passwords with either two lower case letters or two symbols.

(c) Determine the number of possible passwords with either two lower case letters or two upper case letters.

Calculator Assumed

14. [9 marks: 3, 2, 4]

A password consisting of four characters is to be chosen from 26 upper case letters, 26 lower case letters, 10 digits and a set of 32 symbols.

(a) Complete the table below.

Composition of password	Number of different passwords
Exactly 2 lower case letters	
Exactly 1 symbol	
Exactly 2 lower case letters and 1 symbol	

(b) Determine the number of possible passwords with either two lower case letters or one symbol.

(c) Determine the number of possible passwords with either one symbol or one upper case letter.

Calculator Assumed

15. [7 marks: 4, 3]

A sample of ten students is to be selected from a group of four year 8, five year 9, six year 10, seven year 11 and eight year 12 students.

(a) Complete the table below.

Composition of sample	Number of different samples
Five year 12 students.	
Three year 11 students.	
Two year 10 students.	
Five year 12 and three year 11 students.	
Five year 12 and two year 10 students.	
Three year 11 and two year 10 students.	
Five year 12 and three year 11 and two year 10 students.	

(b) Determine the number of possible samples with five year 12 or three year 11 or two year 10 students.

04 Combinatorics IV

Calculator Free

1. [6 marks: 1, 2, 1, 2]

A container has 10 different sized pairs of nuts and bolts with the nuts removed from the respective bolts.

(a) Five bolts are randomly removed from this container. What is the minimum number of nuts that need to be removed from this container to ensure one matching pair of nut and bolt?

(b) Six bolts are randomly removed from this container. What is the minimum number of nuts that need to be removed from this container to ensure two matching pairs of nut and bolt?

(c) What is the minimum number of items that need to be removed from this container to ensure one matching pair of nut and bolt?

(d) Fourteen items are randomly removed from the container.

(i) What is the minimum number of matching pairs of nuts and bolts?

(ii) What is the maximum possible number of matching pairs?

Calculator Free

2. [7 marks: 1, 2, 2, 2]

Twelve dog owners and their dogs (one dog per owner) meet at a dog park.

(a) Six dog owners are randomly chosen. What is the minimum number of dogs that need to be chosen to ensure a matching owner-dog pair?

(b) Seven dogs are randomly chosen. What is the minimum number of owners that need to be chosen to ensure three matching owner-dog pairs?

(c) A total of 15 owners and dogs were randomly selected.

(i) What is the minimum number of owner-dog pairs in this selection?

(ii) What is the maximum possible number of owner-dog pairs in this selection?

(d) If the owners came as couples, that is one dog per couple, what is the minimum number of persons and dogs that need to be chosen to ensure:

(i) a matching owner-dog pair?

(ii) more than two matching owner-dog pairs?

Calculator Free

3. [7 marks: 1 each]

A container has 5 red marbles, 6 green marbles and 9 yellow marbles. What is the minimum number of marbles that need to be drawn from this container to ensure:

(a) a marble of each colour?

(b) two marbles of each colour?

(c) two red marbles?

(d) two yellow marbles?

(e) two green marbles?

(f) two marbles of the same colour?

(g) three marbles of the same colour?

Calculator Free

4. [7 marks: 1 each]

Dennis has 3 blue pens, 4 red pens and 5 black pens in his pencil case. What is the minimum number of pens that need to be drawn from the pencil case to ensure that:

(a) a red pen is drawn?

(b) a pen of each colour is drawn?

(c) a red and a blue pen is drawn?

(d) two blue pens and two red pens are drawn?

(e) two red pens and two black pens are drawn

(f) two blue pens and two black pens are drawn?

(g) two pens of the same colour are drawn?

Calculator Assumed

5. [4 marks: 2, 2]

There are 25 students in a class.

(a) Explain clearly why there must be at least 3 students that are born in a same month.

(b) Explain clearly why there must be at least one month which is a birth month shared by no more than 2 students.

Calculator Assumed

6. [6 marks: 1, 1, 2, 2]

A class has 30 students.

(a) How many students need to be chosen to ensure that there are:
(i) two students who are born on the same day of the week?

(ii) five students who are born on the same day of the week?

(b) There are at least x students who are born on the same day of the week. Find x. Justify your answer.

(c) There must be at least one day of the week which is the birth day of no more than y students. Find y. Justify your answer.

Calculator Assumed

7. [7 marks: 1, 1, 1, 2, 2]

There are 300 students in a primary school.

(a) How many students need to be chosen to ensure that there are:

(i) two students with family names that start with the same letter?

(ii) three students with family names that start with the same letter?

(iii) six students with family names that start with the same letter?

(b) There are at least x students with family names that start with the same letter. Find x. Justify your answer.

(c) There must be at least one letter that is the first letter of the family names of no more than y students. Find y. Justify your answer.

05 Addition & Subtraction of Vectors (Using Trigonometry)

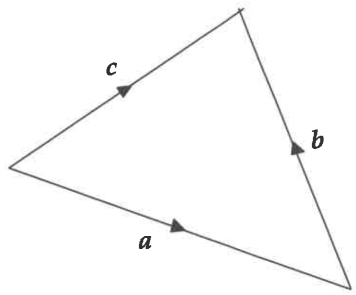
Calculator Free

1. [2 marks: 1, 1]

Vectors ***a***, ***b*** and ***c*** are as drawn in the accompanying diagram.

(a) Express ***c*** in terms of ***a*** and ***b***.

(b) Express ***a*** in terms of ***b*** and ***c***.



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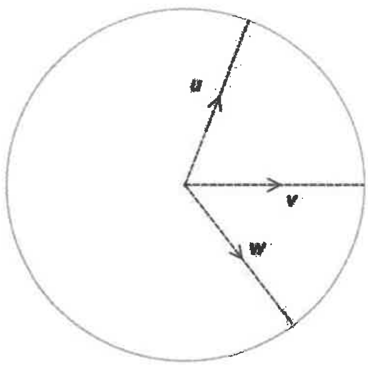
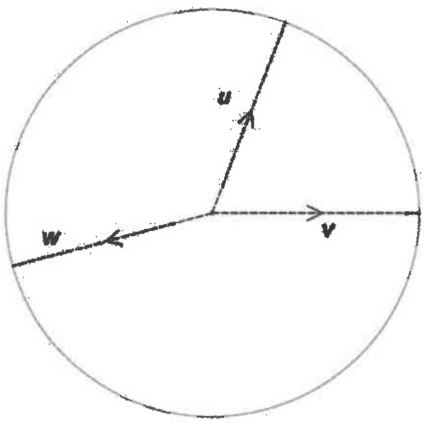
2. [4 marks: 2, 2]

The accompanying diagram shows the relative positions of the vectors ***u***, ***v*** and ***w***. $|\boldsymbol{u}| = |\boldsymbol{v}| = |\boldsymbol{w}|$.

(a) Sketch in the space provided below ***u*** + ***v*** + ***w***.



(b) Sketch in the space provided below ***v*** – ***u*** – ***w***.



Calculator Assumed

3. [7 marks: 3, 4]

The angle between $\boldsymbol{u}$ and $\boldsymbol{u} - \boldsymbol{v}$ is 30°, $|\boldsymbol{v}| = 5$ and $|\boldsymbol{u} - \boldsymbol{v}| = 10$.

(a) Draw a clearly labelled sketch of the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{u} - \boldsymbol{v}$.

(b) Use the rules of trigonometry to find $|\boldsymbol{u}|$ and the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

---

4. [6 marks: 2, 4]

An aircraft is flying with a speed of 400 kmh^{-1} along bearing 145°. The aircraft is buffeted by a strong wind of magnitude 80 kmh^{-1} blowing from bearing 240°.

(a) Draw a sketch to indicate the actual direction of the aircraft.

Calculator Assumed

4. (b) Find the ground speed and direction of the aircraft.

5. [6 marks: 3, 3]

A boy intends to swim across a river of width 20 metres to the opposite bank. The river flows at a steady rate of 1 kmh^{-1}. The boy can swim at a steady speed of 2 kmh^{-1}.

(a) In what direction should the boy be headed so that he ends up at the opposite bank directly opposite to where he started off ?

(b) Find the time taken for the swim in part (a).

Calculator Assumed

6. [5 marks]

A current is flowing in the direction N48°E at 10 kmh^{-1}. With what speed and in what direction should a naval vessel be travelling to achieve a resultant speed of 40 kmh^{-1} in the direction N30°W.

06 Components & Position Vectors I

Calculator Free

1. [8 marks: 2, 1, 1, 4]

   Given that $\boldsymbol{a} = -2\,\mathrm{i} + 6\,\mathrm{j}$ and $\boldsymbol{b} = 5\,\mathrm{i} - 4\,\mathrm{j}$, find:

   (a) $|\boldsymbol{a} + \boldsymbol{b}|$.

   (b) the unit vector parallel to $\boldsymbol{a} + \boldsymbol{b}$.

   (c) a vector that is parallel to $\boldsymbol{a} + \boldsymbol{b}$ but with a magnitude of 5.

   (d) $\boldsymbol{a}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ where $\boldsymbol{p} = 2\mathrm{i} + \mathrm{j}$ and $\boldsymbol{q} = -3\mathrm{i} + 2\mathrm{j}$.

Calculator Free

2. [6 marks]

   $\mathbf{OA} = 3\mathbf{i} + 10\mathbf{j}$, $\mathbf{OB} = 5\mathbf{i} + b\mathbf{j}$ and $\mathbf{OC} = 9\mathbf{i} + c\mathbf{j}$.
   Find c in terms of b if A, B and C are collinear.

---

3. [7 marks]

   Vector $a\mathbf{i} + (a + b)\mathbf{j}$ has a magnitude of 5 and is parallel to vector $4\mathbf{i} + 8\mathbf{j}$.
   Find all possible values of a and b.

Calculator Free

4. [5 marks]

Vector $a\,\mathbf{i} + 10\mathbf{j}$ is of the same magnitude as $(b - 10)\,\mathbf{i} + (a - 2b)\,\mathbf{j}$ but acts in the opposite direction. Find the values of a and b.

5. [4 marks]

Vector $\begin{pmatrix} a \\ b \end{pmatrix}$ has magnitude 20 and is parallel to $\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$.

Find the values of a and b.

6. [3 marks]

The point K divides the line segment AB internally in the ratio 4 : 1. Use a vector method to find the position vector of K if $\mathbf{OB} = -\,\mathbf{i} + 2\,\mathbf{j}$ and $\mathbf{AB} = 15\,\mathbf{i} - 5\,\mathbf{j}$.

Calculator Assumed

7. [3 marks]

The points P and Q have position vectors $5\mathbf{i} - 2\mathbf{j}$ and $-4\mathbf{i} + 5\mathbf{j}$ respectively. The point K is such that $\mathbf{PK} = -4\mathbf{QK}$. Find the position vector of K.

8. [4 marks]

It is known that $\mathbf{OA} = a\mathbf{i} + \mathbf{j}$ and $\mathbf{OB} = 4\mathbf{i} + b\mathbf{j}$.
K is a point such that AK: AB = 2 : 5 and $\mathbf{OK} = 4\mathbf{i} - 3\mathbf{j}$. Find a and b.

Calculator Assumed

9. [5 marks]

Vector $\boldsymbol{u}$ has magnitude 100 kmh^{-1} and acts in the direction 040°.
Vector $\boldsymbol{v}$ has magnitude 150 kmh^{-1} and acts in the direction 280°.
Let $\boldsymbol{i}$ be the unit vector in the West-East direction
and $\boldsymbol{j}$ be the unit vector in the South-North direction.

Use vector components to find the magnitude and direction of $\boldsymbol{u} - 2\boldsymbol{v}$.

---

10. [5 marks]

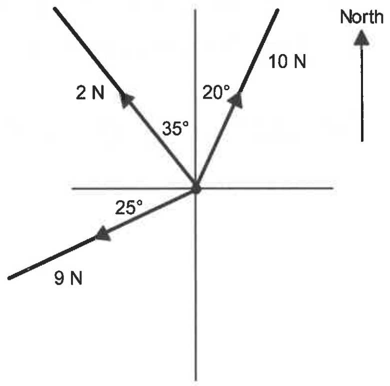
Given that $\boldsymbol{u} = \langle -4, 16 \rangle$ and $|\boldsymbol{v}| = 100$, find $\boldsymbol{v}$
if $\boldsymbol{u} + \boldsymbol{v}$ is to be in the same direction as the vector $\langle 2, 2 \rangle$.

07 Components & Position Vectors II

Calculator Assumed

1. [6 marks: 3, 3]

   The diagram below shows the forces acting on a body. The forces are all on the same plane.



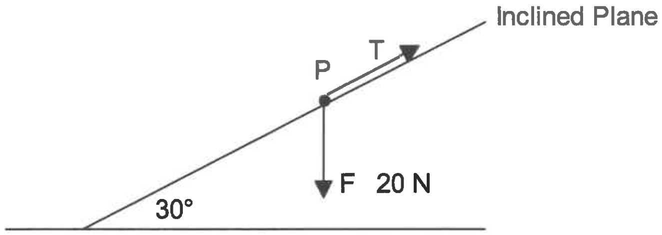
   (a) Find the magnitude of the resultant.

   (b) Find the magnitude and the direction of a single force that will keep this system in equilibrium.

Calculator Assumed

2. [5 marks: 2, 2, 1]

In the diagram below, a particle P is on a plane inclined at an angle of 30° to the horizontal. A vertical force F of magnitude 20 N is acting on P as shown. Force T parallel to the inclined plane is applied to prevent P from slipping down the inclined plane.



(a) Find the magnitude of the component of F parallel to the inclined plane.

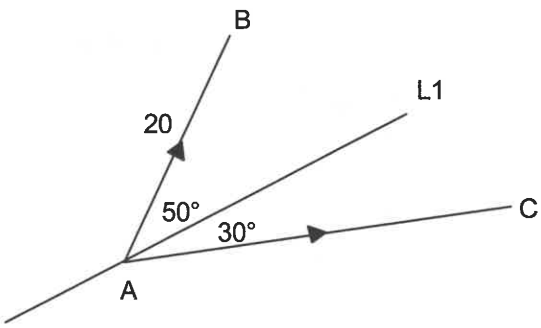
(b) Find the magnitude of the component of F perpendicular to the inclined plane.

(c) Find the magnitude of force T.

Calculator Assumed

3. [4 marks: 1, 1, 2]

In the diagram below, vector **AB** is of magnitude 20 units and is inclined at an angle of 50° to the line L1. Vector **AC** is inclined at angle of 30° to the line L1 as shown.



(a) Find the magnitude of the component of **AB** parallel to the line L1.

(b) Find the magnitude of the component of **AB** perpendicular to line L1.

(c) Find the magnitude of **AC** if the resultant of the vectors **AB** and **AC** is parallel to the line L1.

Calculator Assumed

4. [11 marks: 1, 1, 4, 2, 3]

A light plane can fly at 80 km per hour in still air. The pilot wishes to fly from O to a neighbouring airstrip Q, located 40 km from O in the direction 060°. A constant wind of 20 km per hour is blowing from the North. **i** and **j** are unit vectors in the Easterly and Northerly directions respectively.

(a) Write in terms of **i** and **j** the position vector of Q relative to O.

(b) Write in terms of **i** and **j** the velocity vector of the wind.

(c) Find the velocity vector the pilot should set so that the plane flies directly to Q.

(d) Find the resultant speed of the plane.

(e) Find the difference in flying time (to the nearest minute) caused by the wind.

Calculator Assumed

5. [8 marks: 4, 2, 2]

A helicopter capable of flying at a speed of 100 km per hour in still air, takes off from O for a mining town located at A. The position vector of A relative to O is 200**i** – 300**j** km. Throughout the journey, the helicopter encounters a wind blowing with velocity 13**i** + 5**j** km per hour.

(a) Find the velocity vector the pilot should set so that the helicopter flies directly to A.

(b) Find the time taken for the journey.

(c) Find the velocity vector the pilot should set for a direct flight back to O. Assume that the wind blows with the same velocity throughout the flight back.

08 Components & Position Vectors III

Calculator Assumed

1. [6 marks: 1, 1, 1, 3]

A particle P, initially at $5\mathbf{i} - 10\mathbf{j}$ metres, moves with velocity $3\mathbf{i} + 4\mathbf{j}$ ms^{-1}.

(a) Find the position vector of P after 10 seconds.

(b) Find the distance travelled by P after 10 seconds.

(c) Find the position vector of P after t seconds.

(d) When is P at a point with position vector $(65\mathbf{i} + 70\mathbf{j})$ metres.

Calculator Assumed

2. [9 marks: 3, 3, 3]

The position vector of particles A and B, t hours after 12 noon, are $\mathbf{r} = 12\mathbf{i} + 3\mathbf{j} + t\,(3\mathbf{i} + 4\mathbf{j})$ and $\mathbf{r} = -3\mathbf{i} - 5\mathbf{j} + t\,(2\mathbf{i} + 6\mathbf{j})$ metres respectively.

(a) Find in terms of t, the distance between A and B t hours after 12 noon.

(b) Find when A and B are 18 metres apart.

(c) Find when A is closest to B and find this distance.

Calculator Assumed

3. [9 marks: 2, 1, 2, 4]

The position vectors of A and B, t hours after 10 am are
$\mathbf{r} = -4\mathbf{i} - 4\mathbf{j} + t\,(2\mathbf{i} + 3\mathbf{j})$ and $\mathbf{r} = 3\mathbf{i} + 10\mathbf{j} + t\,(a\,\mathbf{i} + \mathbf{j})$ respectively.

(a) Find **AB** t hours after 10 am.

(b) Find in terms of a and t, the distance between A and B, t hours after 10 am.

(c) Explain why when collision between A and B occurs, **AB** = $0\mathbf{i} + 0\mathbf{j}$

(d) Find the value of a if the two particles never collide.

Calculator Assumed

4. [12 marks: 2, 4, 2, 4]

At 9.00 am, boat A is located at $2\mathbf{i} + 3\mathbf{j}$ km and is travelling with velocity $2\mathbf{i} - 3\mathbf{j}$ km per hour. At 10.00 am, boat B is located at $2\mathbf{i} + 3\mathbf{j}$ km and is travelling with a speed of 5 km per hour. Let the velocity of boat B be $x\mathbf{i} + y\mathbf{j}$ km per hour.

(a) Find the position vectors of A and B t hours after 10.00 am.

(b) Show that when B intercepts A, $y = -1.5x$.

(c) Hence, find the direction B should take in order to intercept A.

(d) Find when B intercepts A.

Calculator Assumed

5. [6 marks: 1, 1, 2, 2]

Particle P starts moving from the point A with position vector $-2\mathbf{i} + 3\mathbf{j}$ metres with velocity $\mathbf{i} - 2\mathbf{j}$ metres per second. Particle Q starts moving from the point A at the same time with velocity $2\mathbf{i} + 3\mathbf{j}$ metres per second.

(a) Determine the position vector of P after t seconds.

(b) Determine the position vector of Q after t seconds.

(c) Find in terms of t, the distance between P and Q after t seconds.

(d) Use your answer in (c) to find when P and Q are 10 metres apart.

Calculator Assumed

6. [7 marks: 1, 2, 1, 3]

Particle P starts moving from the point with position vector 3**i** + 5**j** metres with velocity 2i – 3j metres per second.

(a) Determine the position vector of P after 3 seconds.

(b) Find the distance from P to the point with position vector –2**i** + **j** after 3 seconds.

(c) Determine the position vector of P after t seconds.

(d) Find when P is closest to the origin and state this distance.

Calculator Assumed

7. [9 marks: 2, 2, 2, 3]

A speed boat is moving at a constant velocity of 40 kmh^{-1} in the direction with bearing 060°. Initially, the speed boat is $5\sqrt{2}$ km from a buoy and is in the direction with bearing 225° from the buoy. Given that **i** and **j** are the unit vectors in the Easterly direction and Northerly direction respectively, find:

(a) the initial position vector of the speed boat with respect to the buoy in terms of **i** and **j**.

(b) the direction vector of the speed boat in terms of **i** and **j**.

(c) the position vector of the speed boat with respect to the buoy t hours later.

(d) the time when the speed boat is nearest to the buoy and the least distance between the speed boat and the buoy.

Calculator Assumed

8. [9 marks: 4, 2, 3]

Yacht A starts sailing from the point L with position vector 5**i** – 2**j** metres with velocity –**i** + 3**j** . Yacht B starts sailing 10 seconds later with velocity 2**i** + **j** metres per second, from the point M with position vector 4**i** – 3**j** metres per second. t is time in seconds from the moment B starts sailing.

(a) Find in terms of t, the distance between A and B after t seconds.

(b) When will A and B be 40 metres apart?

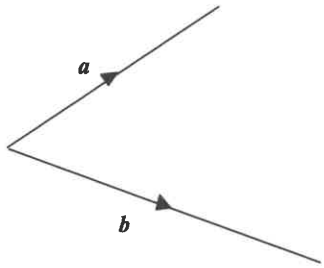
(c) When will the two yachts be closest together? State this distance.

09 Relative Displacement

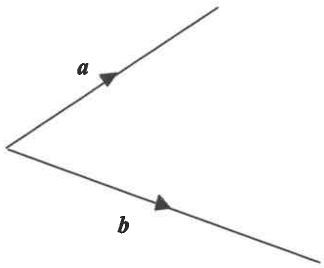
Calculator Free

1. [4 marks: 1, 1, 1, 1]

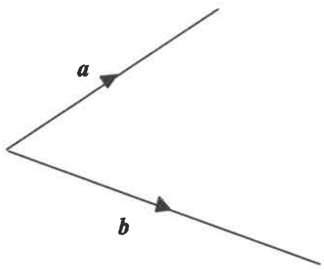
(a) Indicate clearly in the diagram below $\boldsymbol{a} + \boldsymbol{b}$.



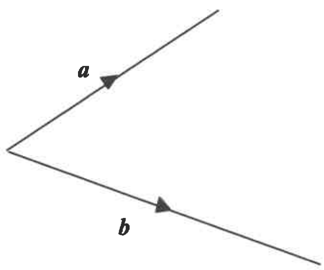
(b) Indicate clearly in the diagram below $\boldsymbol{a} - \boldsymbol{b}$.



(c) Indicate clearly in the diagram below the vector $\boldsymbol{b}$ relative to $\boldsymbol{a}$.



(d) Indicate clearly in the diagram below the vector $\boldsymbol{a}$ relative to $\boldsymbol{b}$.



Calculator Free

2. [4 marks: 1, 1, 2]

The position vectors of the points A, B, and C with respect to the origin O are $\mathbf{i} + \mathbf{j}$, $2\mathbf{i} - \mathbf{j}$ and $-4\mathbf{i} + 5\mathbf{j}$ respectively.

(a) Find the position vector of A relative to B.

(b) Find the displacement of C relative to A.

(c) The position vector of the point D relative to C is $-2\mathbf{i} - 3\mathbf{j}$.
Find the position vector of D relative to O.

---

3. [2 marks: 1, 1]

The position vectors of the yachts WAone and WAtwo are $20\mathbf{i} + 40\mathbf{j}$ km and $-10\mathbf{i} + 50\mathbf{j}$ km respectively.

(a) Find the position vector of WAtwo relative to WAone.

(b) Hence, find the distance between the two yachts.

Calculator Assumed

4. [5 marks: 2, 1, 2]

The position vector of Peter relative to a flag pole is 20**i** + 40**j** metres. Relative to Peter, Joe has position vector 5**i** – 15**j** metres.
(a) Find the position vector of Joe relative to the flagpole.

(b) Hence, find the distance between Joe and the flag pole.

(c) The position vector of Kelly relative to Joe is a**i** + 20**j** metres.
Find the value of a if the distance between Kelly and Joe is 50 metres.

---

5. [4 marks]

Vectortown has position vector –20i + 10j km. The position vector of Trigtown relative to Vectortown is –40i – 15j km. The position vector of Easytown relative to Trigtown is 4i + 70j km. Find the position vector of Easytown.

10 Relative Velocity

Calculator Assumed

1. [7 marks: 1, 1, 3, 2]

Relative to an observer at O, A is moving with velocity $6\mathbf{i} + 9\mathbf{j}$ ms^{-1} and B is moving with velocity $-3\mathbf{i} + 4\mathbf{j}$ ms^{-1}.

(a) Find the velocity of A relative to B

(b) What is the speed of A relative to B?

(c) The velocity of C relative to B is $4\mathbf{i} - 5\mathbf{j}$ ms^{-1}. Find the velocity of C relative to A.

(d) In what direction is C moving relative to A?

Calculator Assumed

2. [7 marks: 3, 4]

L is travelling along bearing 060° with a speed of 10 kmh^{-1}. M is travelling along bearing 210° with a speed of 5 kmh^{-1}.

(a) Draw a clearly labelled vector diagram indicating the velocity vector of L relative to M.

(b) Use trigonometry to find the speed and direction of L relative to M.

Calculator Assumed

3. [7 marks: 2, 2, 1, 2]

P is travelling along bearing 045° with a speed of 20 kmh^{-1}. Q is travelling along bearing 240° with a speed of 15 kmh^{-1}. **i** and **j** are unit vectors in the Easterly and Northerly directions respectively.

(a) Express the velocities of P and Q in terms of **i** and **j**.

(b) Find the velocity of Q relative to P.

(c) What is the speed of Q relative to P?

(d) What is the direction of Q relative to P?

11 Relative Vectors

Calculator Assumed

1. [5 marks: 2, 3]

James is running along bearing 050° with a speed of 5 ms^{-1}. Wesley is running along bearing 300° with speed 4 ms^{-1}. **i** and **j** are unit vectors in the Easterly and Northerly directions respectively.

(a) Find in component form the velocity of Wesley relative to James.

(b) Find how fast and in what direction is Wesley moving away from James.

Calculator Assumed

2. [6 marks: 2, 4]

A yacht is sailing on a bearing of 050° with speed 12 kmh^{-1}. Sarah on the yacht measures the wind as blowing with a speed of 10 kmh^{-1} from a bearing of 300°.

(a) Sketch a clearly labelled velocity vector diagram that shows the relationship between the velocity of the yacht, the true velocity of the wind and the velocity of wind relative to the yacht.

(b) Find the true speed and direction of the wind.

Calculator Assumed

3. [7 marks: 2, 2, 3]

A yacht Y is moving with velocity $\mathbf{2i} + \mathbf{5j}$ kmh^{-1} . A sailor on board the yacht measures the wind as blowing with velocity $-3\mathbf{i} - 2\mathbf{j}$ kmh^{-1}.

(a) Find the velocity of the wind.

To a sailor on a second yacht Z, the wind appears to be blowing with velocity $2\mathbf{i} + 4\mathbf{j}$ kmh^{-1}.

(b) Find the velocity of the second yacht.

(c) How fast is yacht Z moving away from yacht Y and in what direction?

Calculator Assumed

4. [4 marks]

A ship is travelling with a speed of 20 knots along bearing 080°. Relative to the ship, the wind is blowing from 310° with a speed of 8 knots. Let **i** and **j** be unit vectors in the Easterly and Northerly directions respectively. By expressing the given velocities in component form, find the true speed and direction of the wind.

Calculator Assumed

5. [11 marks: 2, 2, 7]

May is running along 025° at 4 kmh^{-1}.
Relative to May, Fay is running with speed a kmh^{-1} along bearing 150°.

(a) Find in terms of a, the velocity of Fay, v_f.

Jane is running due East at 2 kmh^{-1}.
Relative to Jane, Fay is running with speed b kmh^{-1} along bearing 120°.

(b) Find in terms of b, the velocity of Fay, v_f.

(c) Use your answers in (a) & (b) to find a and b. Hence, find the true speed and direction with which Fay is running.

Calculator Assumed

6. [9 marks: 3, 3, 3]

Tom is in a hot-air balloon travelling with velocity $3\mathbf{i} + 4\mathbf{j}$ km and to Tom the wind is blowing along bearing 045°. Jerry is in another hot-air balloon flying at the same altitude and travelling with velocity $-4\mathbf{i} + \mathbf{j}$. To Jerry, the wind is blowing along bearing 60°. Let the true velocity of the wind be $x\mathbf{i} + y\mathbf{j}$ kmh^{-1}. **i** and **j** are unit vectors in the Easterly and Northerly directions respectively.

(a) Find in terms of x and y, the velocity of the wind relative to Tom. Hence, show that $x - y = -1$.

(b) Find in terms of x and y, the velocity of the wind relative to Jerry. Hence, show that $x - \sqrt{3}y = -4 - \sqrt{3}$.

(c) Find the true velocity of the wind.

Calculator Assumed

7. [4 marks: 1, 1, 2]

At 1000 hours, the position vectors of A and B are $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. A is moving with velocity $\boldsymbol{u}$ and B is moving with velocity $\boldsymbol{v}$.

(a) Find ${}_{\mathbf{B}}\boldsymbol{r}_{\mathbf{A}}$ the position vector of B relative to A.

(b) Find ${}_{\mathbf{A}}\boldsymbol{v}_{\mathbf{B}}$ the velocity of A relative to B.

(c) Show that if A collides with B after t seconds, ${}_{\mathbf{B}}\boldsymbol{r}_{\mathbf{A}} = t\, {}_{\mathbf{A}}\boldsymbol{v}_{\mathbf{B}}$ where $t > 0$.

---

8. [4 marks: 3, 1]

At 0800 hours, the position vector of Q relative to P is $45\mathbf{i} - 60\mathbf{j}$ km. The velocity of P relative to Q is a constant $9\mathbf{i} + m\,\mathbf{j}$ kmh^{-1}.

(a) Find the value of m, if P intercepts Q.

(b) Find when P intercepts Q.

Calculator Assumed

9. [6 marks: 1, 1, 4]

At 0900 hours, P is located at $20\mathbf{i} - 10\mathbf{j}$ km and is travelling with a constant velocity of $-4\mathbf{i} + 6\mathbf{j}$ kmh^{-1}. At the same time, Q is located at $50\mathbf{i} + 40\mathbf{j}$ km and is travelling with a constant velocity of $-10\mathbf{i} - 4\mathbf{j}$ kmh^{-1} .

(a) Find the displacement of Q relative to P at 0900 hours.

(b) Find the velocity of P relative to Q at 0900 hours.

(c) Use your answers in (a) and (b) to determine when and where P will collide with Q.

Calculator Assumed

10. [11 marks: 2, 2, 5, 2]

When the clock struck one, relative to a clock tower, the position vector of a cat is $6\mathbf{i} + 10\mathbf{j}$ m. The cat is running with velocity $2\mathbf{i} + 2\mathbf{j}$ ms^{-1}. At the same time, relative to the same clock tower, the position vector of a dog is $-4\mathbf{i} - 6\mathbf{j}$. The dog can run at 5 ms^{-1}.

(a) Find the position vector of the cat relative to the dog.

Let the velocity vector of the dog for it to intercept the cat be $x\mathbf{i} + y\mathbf{j}$ ms^{-1}.

(b) Find in terms of x and y, the velocity of the dog relative to the cat.

(c) Use your answers in (a) and (b) to find x and y for the dog to intercept the cat. Assume that the cat continues running with the same speed and in the same direction.

(d) When and where does the interception take place.

Calculator Assumed

11. [16 marks: 1, 1, 3, 1, 4, 3, 3]

The locations and velocities of three boats at 0900 hours are given in the table below. Assume that each boat moves with constant velocity.

Boat	Location (km)	Velocity (kmh^{-1})
A	$20\mathbf{i} + 40\mathbf{j}$	$-4\mathbf{i} - 6\mathbf{j}$
B	$-25\mathbf{i} + 20\mathbf{j}$	$5\mathbf{i} + -2\mathbf{j}$
C	$35\mathbf{i} - 10\mathbf{j}$	$-7\mathbf{i} + 4\mathbf{j}$

(a) Find the position vector of A relative to B at 0900 hours.

(b) Find the velocity of B relative to A.

(c) Use your answer in (b) to determine at what time B will meet A.

(d) Determine the position vector of the point where A will meet B.

Calculator Assumed

11. (e) Find the distance between A and C t hours after 0900 hours.

(f) Find the time when C will meet A.

(g) Prove that all three boats will arrive at a similar spot at the same time.

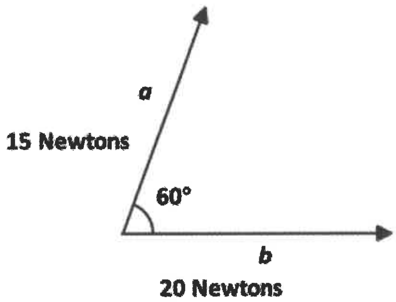
12 Scalar Product I

Calculator Free

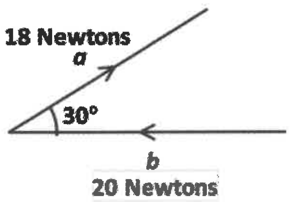
1. [4 marks: 2, 2]

Find the scalar product between the vectors ***a*** and ***b*** as given:

(a)



(b)



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2. [6 marks: 2, 2, 2]

Given that vector ***u*** has magnitude 10 ms^{-1} in the direction 030°, ***v*** has magnitude 15 ms^{-1} in the direction 090° and ***w*** has magnitude 5 ms^{-1} in the direction 180°. Find:

(a) ***u*** . ***v***

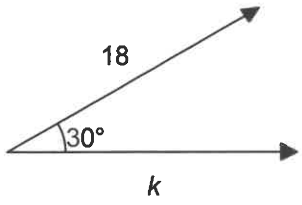
(b) ***u*** . ***w***

(c) the magnitude and direction of (***u*** . ***w***) ***v***.

Calculator Free

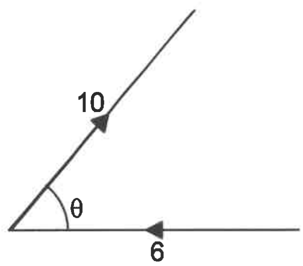
3. [2 marks]

The scalar product between the two vectors shown is $180\sqrt{3}$. Find *k*.



4. [2 marks]

The scalar product between the two vectors shown is –50. Find cos θ.



5. [5 marks]

Given that $|\boldsymbol{a}| = 20$ and $|\boldsymbol{b}| = 25$, find with reasons the maximum and minimum value of $\boldsymbol{a} \cdot \boldsymbol{b}$.

Calculator Free

6. [4 marks: 2, 1, 1]

Given that $|\boldsymbol{a}| = 8$ and $|\boldsymbol{b}| = 5$, and if $\boldsymbol{a} \cdot \boldsymbol{b} = 20\sqrt{2}$, find θ, the acute angle between:

(a) $\boldsymbol{a}$ and $\boldsymbol{b}$

(b) $2\boldsymbol{a}$ and $3\boldsymbol{b}$

(c) $\boldsymbol{a}$ and $-\boldsymbol{b}$.

---

7. [6 marks: 3, 3]

Given that $|\boldsymbol{m}| = 10$ and $|\boldsymbol{n}| = 10$, find $\boldsymbol{n}$ in terms of $\boldsymbol{m}$ if:

(a) $\boldsymbol{m} \cdot \boldsymbol{n} = 100$

(b) $\boldsymbol{m} \cdot \boldsymbol{n} = -100$

Calculator Free

8. [4 marks: 1, 1, 2]

Given that $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find:

(a) $(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c}$

(b) $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$

(c) k if $\boldsymbol{a} \cdot (3\mathbf{i} + k\mathbf{j}) = 9$

---

9. [3 marks: 1, 1, 1]

Expand and simplify:

(a) $(\boldsymbol{m} + \boldsymbol{n}) \cdot (\boldsymbol{m} + \boldsymbol{n})$

(b) $(\boldsymbol{c} + \boldsymbol{d}) \cdot (\boldsymbol{c} - \boldsymbol{d})$

(c) $(\boldsymbol{m} - 3\boldsymbol{n}) \cdot (2\boldsymbol{m} - \boldsymbol{n})$

Calculator Free

10. [4 marks: 2, 2]

Given that $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular, prove that:

(a) $\boldsymbol{p} \cdot \boldsymbol{q} = 0$

(b) $(\boldsymbol{p} + \boldsymbol{q}) \cdot (\boldsymbol{p} + \boldsymbol{q}) = |\boldsymbol{p}|^2 + |\boldsymbol{q}|^2$

---

11. [7 marks: 2, 2, 3]

Given that $|\boldsymbol{r}| = 10$ and $|\boldsymbol{s}| = 8$, find:

(a) $\boldsymbol{r} \cdot \boldsymbol{r}$

(b) $(\boldsymbol{r} + \boldsymbol{s}) \cdot (\boldsymbol{r} + \boldsymbol{s})$ if $\boldsymbol{r}$ and $\boldsymbol{s}$ are parallel and in the same direction.

(c) $|\boldsymbol{r} - \boldsymbol{s}|$ if $\boldsymbol{r}$ and $\boldsymbol{s}$ are perpendicular.

Calculator Assumed

12. [7 marks: 2, 3, 2]

Given that $\boldsymbol{u} = \mathbf{i} + 3\mathbf{j}$, $\boldsymbol{v} = -6\mathbf{i} + 8\mathbf{j}$ and $\boldsymbol{w} = k\mathbf{i} - 2\mathbf{j}$, find:

(a) the acute angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(b) k, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel in the opposite direction.

(c) k, if the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is $120°$.

Calculator Assumed

13. [6 marks: 3, 3]

Given that $|\boldsymbol{u}| = 10$ and $(\boldsymbol{u} + 2\boldsymbol{v}) \cdot (\boldsymbol{u} + \boldsymbol{v}) = 408$.

(a) Find $|\boldsymbol{v}|$ if $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular.

(b) Find $|\boldsymbol{v}|$ if $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel and in opposite directions.

---

14. [6 marks: 3, 3]

Given that $\boldsymbol{u} = -\mathbf{i} + \mathbf{j}$, find in exact form, a unit vector $\hat{\boldsymbol{v}}$, if:

(a) $\boldsymbol{u}$ is perpendicular to $\hat{\boldsymbol{v}}$.

(b) the acute angle between $\boldsymbol{u}$ and $\hat{\boldsymbol{v}}$ is 45°.

Calculator Assumed

15. [6 marks]

Given that $\boldsymbol{u} = 3\mathbf{i} + \mathbf{j}$ and $\boldsymbol{v} = x\mathbf{i} + y\mathbf{j}$, find x and y if $|\boldsymbol{v}| = \sqrt{10}$ and the acute angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is 60°.

13 Scalar Product II

Calculator Free

1. [9 marks: 1, 2, 2, 2, 2]

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have magnitudes 3 and 4 respectively. The acute angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\cos^{-1}\frac{1}{3}$. Find in terms of the vectors $\boldsymbol{a}$ and/or $\boldsymbol{b}$:

(a) the scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

(b) the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

(c) the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$.

(d) $\boldsymbol{v}$, the component of $\boldsymbol{a}$ that is perpendicular to $\boldsymbol{b}$.

(e) the magnitude of the $\boldsymbol{v}$, the component of $\boldsymbol{a}$ that is perpendicular to $\boldsymbol{b}$.

Calculator Free

2. [10 marks: 2, 2, 3, 3]

(a) The vectors ***a*** and ***b*** have magnitudes 8 and 6 respectively. Find the vector projection of ***a*** onto ***b*** if the angle between the vectors ***a*** and ***b***:

(i) is 60°.

(ii) is 120°.

(b) Find θ, the acute angle between ***a*** and ***b*** if $|\boldsymbol{a}| = 8$ and the vector projection of ***a*** onto ***b*** has a magnitude of $4\sqrt{3}$.

(c) The vector ***b*** has magnitude 10. The projection of vector ***a*** on ***b*** is 5***b***.

Find the magnitude of ***a*** if the angle between ***a*** and ***b*** is 60°.

Calculator Free

3. [4 marks: 2, 2]

Given $\boldsymbol{a} = < 5, 10 >$ and $\boldsymbol{b} = < 4, 3 >$.

(a) Find the component of $\boldsymbol{a}$ that is parallel to $\boldsymbol{b}$.

(b) Find the component of $\boldsymbol{a}$ that is perpendicular to $\boldsymbol{b}$.

---

4. [6 marks: 2, 2, 2]

Given that $\boldsymbol{a} = < 2, 5 > + < 10, -4 >$, find:

(a) the vector projection of $\boldsymbol{a}$ onto $< 6, 15 >$.

(b) the vector projection of $\boldsymbol{a}$ onto $< -5, 2 >$

(c) the vector projection of $\boldsymbol{a}$ onto $< -5, 0 >$

Calculator Free

5. [11 marks: 3, 2, 2, 4]

The acute angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is 60°. The vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is $< 4, -3 >$ and $|\boldsymbol{v}| = 10$.

(a) Find $\boldsymbol{v}$.

(b) Explain clearly why $\boldsymbol{u} = < 4, -3 > + \lambda < 3, 4 >$.

(c) Find $|\boldsymbol{u}|$.

(d) Find $\boldsymbol{u}$.

Calculator Assumed

6. [11 marks: 1, 3, 2, 3, 2]

Let $u = <2, 1>$, $v = <1, -2>$ and $w = <3, 9>$.

(a) Show that u and v are perpendicular.

(b) Given that $w = mu + nv$, find m and n.

(c) Find the vector projection of w onto $<-2, -1>$.

(d) w is the vector projection of $\lambda <2, 1>$ onto w. Find λ.

(e) If w is the vector projection of $\lambda <2, 1>$ onto w, find the component of $\lambda <2, 1>$ that is perpendicular to w.

Calculator Assumed

7. [7 marks: 5, 2]

The vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is $\frac{1}{2}\boldsymbol{b}$. The vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\boldsymbol{a}$.

(a) Show mathematically that $|\boldsymbol{b}|^2 = 2|\boldsymbol{a}|^2$.

(b) Find the acute angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

14 Scalar Product III

Calculator Assumed

1. [6 marks: 2, 1, 3]

Given that vector $\boldsymbol{a}$ has magnitude 2 ms^{-1} in the direction 060°,
$\boldsymbol{b}$ has magnitude 4 ms^{-1} in the direction 045° :

(a) Find $\boldsymbol{a}$ and $\boldsymbol{b}$ in the form $x\,\boldsymbol{i} + y\,\boldsymbol{j}$.

(b) Find $\boldsymbol{a} \bullet \boldsymbol{b}$.

(c) Use your answers in (a) and/or (b) to find cos 15° in exact form.

Calculator Assumed

2. [6 marks: 1, 5]

A 12 noon, A is at the point with position vector $3\mathbf{i} + 2\mathbf{j}$ km and moving with velocity $4\mathbf{i} - 2\mathbf{j}$ kmh^{-1}. B is a stationary object at the point with position vector $3\mathbf{i} - \mathbf{j}$ km.

(a) Find the position vector of A at time t hours after 12 noon.

(b) Use a scalar product method to find when A is closest to B. State this distance.

Calculator Assumed

3. [8 marks: 1, 1, 2, 4]

At 0600 hours, P is at the point with position vector $-10\mathbf{i} - 20\mathbf{j}$ km and moving with velocity $12\mathbf{i} + 2\mathbf{j}$ kmh^{-1}. At the same instant, Q is at a point with position vector $2\mathbf{i} + 10\mathbf{j}$ km and moving with velocity $10\mathbf{i} - 7\mathbf{j}$ kmh^{-1}.

(a) Find the position vector of P relative to Q at 0600 hours.

(b) Find the velocity of P relative to Q.

(c) Find the position vector of P relative to Q t hours after 0600 hours.

(d) Use your answers in (b) and (c) to find the closest distance between P and Q. State when this occurs.

Calculator Assumed

4. [10 marks: 2, 3, 1, 4]

At 0800 hours, P is at the point with position vector $10\boldsymbol{i} + 15\boldsymbol{j}$ km and moving with constant velocity $\boldsymbol{v}$ kmh^{-1}. At 0900 hours, Q starts moving from a point with position vector $-10\boldsymbol{i} - 50\boldsymbol{j}$ km with constant velocity $3\boldsymbol{i} + 8\boldsymbol{j}$ km h^{-1}.

(a) Find $\boldsymbol{v}$, if P arrives at $4\boldsymbol{i} + 0\boldsymbol{j}$ at 1100 hours.

(b) Find the position vector of P relative to Q, t hours after 0900 hours.

(c) Determine the velocity of P relative to Q.

(d) Use your answers in (b) and (c) to find the closest distance between P and Q. State when this occurs.

Calculator Assumed

5. [10 marks: 2, 2, 1, 5]

A force F_1 of $< 4, 4 >$ Newtons is applied to a body and causes the body to be displaced by $< 3, 1 >$ metres.

(a) Find the component of the applied force along the direction of motion.

(b) Find the component of the applied force perpendicular to the direction of motion.

(c) Determine the work done by the applied force.

(d) Another force F_2 of $< x, y >$ Newtons is applied to the same body. But half as much work is required to cause the same displacement to the body.

(i) Find a possible pair of values for x and y.

(ii) Find x and y if $|F_2| = 2\sqrt{2}$ Newtons.

Calculator Assumed

6. [8 marks: 2, 1, 5]

Two forces < 1, 4 > Newtons and < 3, –2 > Newtons act on a single body P and causes the body P to be displaced by < 8, 4 >.

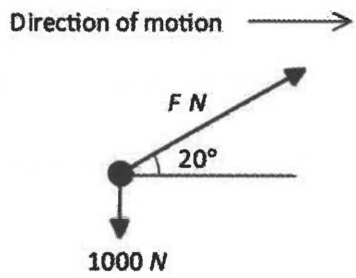
(a) Find the vector projection of the resultant force onto the displacement of P.

(b) Find the work done on P.

(c) A third force of < –2, 1 > Newtons is applied to P and causes it to move by 10 metres. The work done is 36 Joules. Find the displacement vector.

Calculator Assumed

7. [6 marks: 3, 2, 1]



An object of weight 1 000 Newtons is being pulled along a horizontal surface by a force of magnitude F Newtons inclined at an angle of 20° to the surface. The motion of the object is opposed by a horizontal force of magnitude 500 N.

(a) Find in terms of F, the magnitude of the pulling force perpendicular to the surface. Hence, find F.

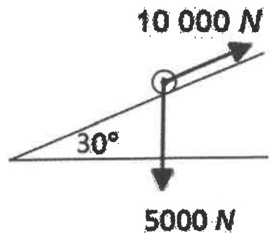
(b) Find the component of the pulling force in the direction of motion. Hence, find the magnitude of the resultant force in the direction of motion.

(c) Find the work done by the pulling force in moving the object 50 m in its direction of motion.

Calculator Assumed

8. [7 marks: 2, 1, 2, 2]

A body of weight 5 000 Newtons is pulled up along a plane inclined at an angle of 30° with the horizontal by a force of magnitude 10 000 N. There is a force of magnitude 500 N opposing the motion of the body.



(a) Find the component of the gravitational force on the body along the inclined plane. Hence, find the magnitude of the resultant force in the direction of the motion of the body.

(b) Find the work done by the resultant force when the body has moved 10 m along the inclined plane.

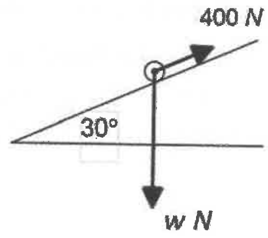
(c) Find the work done by the resultant force when the body has ascended a vertical distance of 10 m.

(d) The work done by the resultant force in moving the body up the inclined plane so that the change in its horizontal position is x metres is 280 kJ. Find x.

Calculator Assumed

9. [7 marks: 3, 4]

A cart of weight w N is at rest on a set of rail-tracks inclined at an angle of 30° with the horizontal. A force parallel to the inclined plane of magnitude 400 N just prevents the body from slipping down the rail-tracks.



(a) Find the component of the gravitational force acting on the cart along the inclined rail-tracks. Hence, find the weight of the cart.

(b) A force of magnitude 1 000 N at an angle of 30° to the inclined rail-tracks acts on the cart and moves it a distance of 10 m along the tracks. Assume that the cart does not leave the rail-tracks. Find the work done by the resultant force along the inclined plane.

15 Geometric Proofs using Vectors

Calculator Assumed

1. [6 marks: 2, 2, 2,]

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-parallel vectors. Find α and β if:

(a) $2\boldsymbol{a} + (\beta - 2)\,\boldsymbol{b} = (1 - \alpha)\,\boldsymbol{a}$

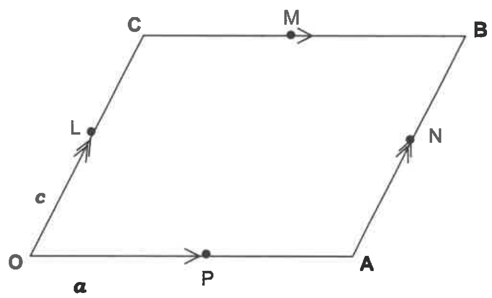
(b) $\alpha\,(3\boldsymbol{a} - 4\boldsymbol{b}) = 6\boldsymbol{a} + \beta\boldsymbol{b}$

(c) $\alpha\boldsymbol{a} + 5\boldsymbol{b}$ is parallel to $3\boldsymbol{a} + \beta\boldsymbol{b}$

---

2. [4 marks: 1, 1, 2]

OABC is a parallelogram. **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$. L, M, N and P are the midpoints of OC, CB, BA and AO respectively.



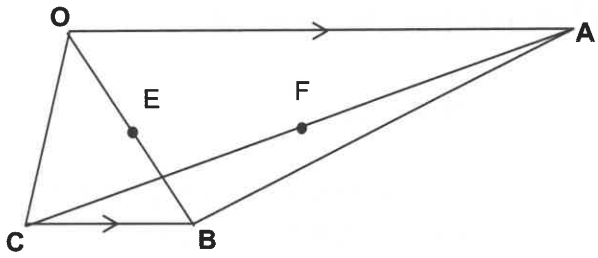
(a) Find **LM** in terms of **a** and **c** .

(b) Find **PN** in terms of **a** and **c**.

(c) Hence, use a vector method to show that LMNP is a parallelogram.

Calculator Assumed

3. [8 marks: 2, 4, 2]



OABC is a trapezium with **OA** = 3**CB**. **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$. E and F are midpoints of OB and CA respectively.

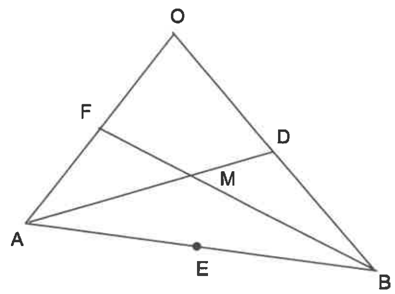
(a) Find **OE** in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

(b) Find **EF** in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

(c) Prove that CEFB is a parallelogram.

Calculator Assumed

4. [14 marks: 2, 2, 5, 3, 2]



OAB is a triangle with **OA** = ***a*** and **OB** = ***b***. D, E and F are the midpoints of OB, AB, and OA respectively. **AM** = α**AD** and **MF** = β**BF**.

(a) Find **AD** and **BF** in terms of ***a*** and ***b***.

(b) Find **AM** and **MF** in terms of ***a*** and ***b***.

(c) Use your answers in (b) to find α and β.

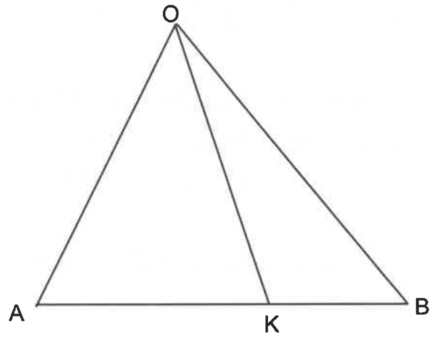
(d) Show that **OM** = μ**OE** giving the value of μ .

(e) Comment on the significance of the location of M in terms of the lines OE, AD and BF.

Calculator Assumed

5. [8 marks: 2, 4, 2]

In triangle OAB, K divides AB in the ratio $\lambda:\mu$ (that is AK:KB = $\lambda:\mu$).



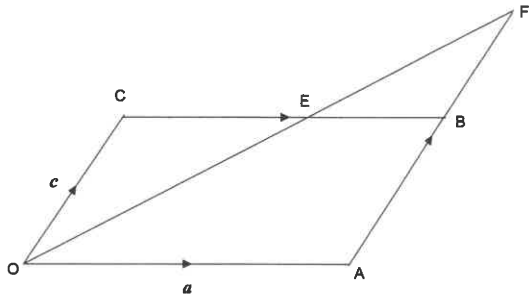
(a) Find **AK** in terms of **OA** and **OB** .

(b) Hence, or otherwise. prove that **OK** = $\left[\dfrac{1}{\lambda+\mu}\right][\lambda\mathbf{OB} + \mu\mathbf{OA}]$.

(c) Use the result above to find the position vector of a point that divides the line connecting A (1, 2) to B (6, 12) in the ratio 2: 3 .

Calculator Assumed

6. [8 marks]



OABC is a parallelogram. The point E divides CB in the ratio $\alpha : \beta$. That is, the point E is such that **EB** = $\dfrac{\beta}{\alpha+\beta}$**CB**. OE extended meets the AB extended at F. Use vector methods to prove that:

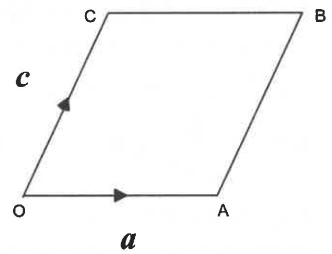
Area of ΔFEB = $\left(\dfrac{\beta}{\alpha+\beta}\right)^2$ × Area of ΔFOA.

[Hint: Let **EF** = λ**OF** and **BF** = μ**AF**.]

Calculator Assumed

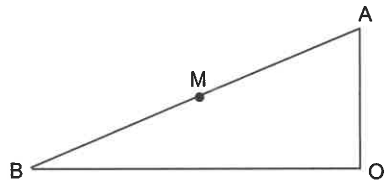
7. [4 marks]

OABC is a rhombus. **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$.
Use a vector method to show that the diagonals of a rhombus are perpendicular to each other.



8. [8 marks: 1, 3, 4]

OAB is a right angled triangle with ∠ AOB = 90°.
OA = $\boldsymbol{a}$ and **OB** = $\boldsymbol{b}$. M is the midpoint of AB.
(a) Explain why $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.



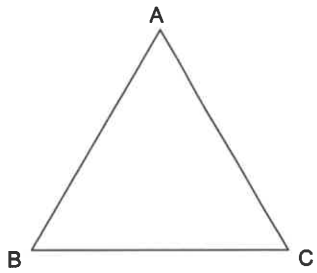
(b) Find $|\mathbf{BM}|^2$ in terms of a and b, where $|\boldsymbol{a}| = a$ and $|\boldsymbol{b}| = b$.

(c) Hence, prove that M is the centre of a circle passing through A, B and O.

Calculator Assumed

9. [12 marks: 2, 3, 4, 3]

ABC is an isosceles triangle with AB = AC.
Also, **BA** = $\boldsymbol{a}$ and **CB** = $\boldsymbol{b}$.



(a) Show that $|\boldsymbol{a}+\boldsymbol{b}| = |\boldsymbol{a}|$

(b) Show that $|\boldsymbol{b}|^2 = -2\,\boldsymbol{a}\,.\,\boldsymbol{b}$.

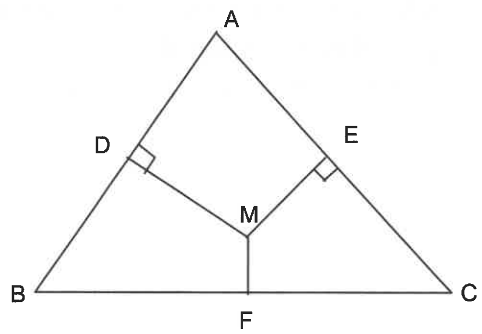
(c) Show that $\cos C = \dfrac{-\boldsymbol{a}\,.\,\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$.

(d) Hence, using a vector method, prove that the base angles of an isosceles triangle are equal.

Calculator Assumed

10. [10 marks: 1, 2, 2, 3, 2]

DM and EM are respectively the perpendicular bisectors of sides AB and AC of triangle ABC. F is midpoint of BC. Also, **AB** = ***b*** , **AC** = ***c*** and **MD** = ***d***.



(a) Find **ME** in terms of ***b***, ***c*** and ***d***.

(b) Use your answer in (a) to show that $[\boldsymbol{d} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b})] \cdot \boldsymbol{c} = 0$

(c) Find **MF** in terms of ***b***, ***c*** and ***d***.

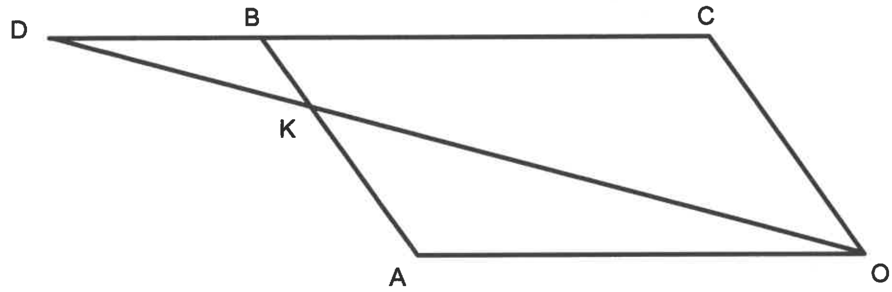
(d) Show that **MF . BC** = 0.

(e) State the significance of the result **MF . BC** = 0 .

Calculator Assumed

11. [5 marks: 2, 3] [TISC]

OABC is a parallelogram with **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$.
The point K divides AB in the ratio 2 : 1. OK extended meets the line CB extended at D. **OK** = α**OD** and **CD** = β**CB**.



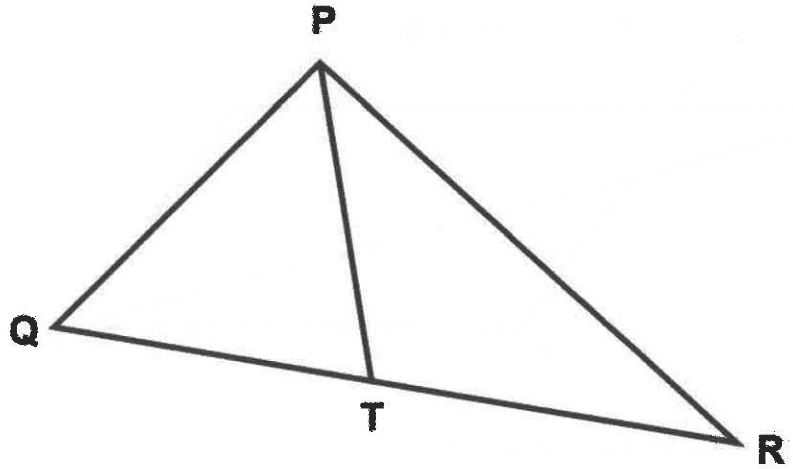
(a) Find **AK** and **OK** in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

(b) Use vector methods to prove that B divides the line CD in the ratio 2 : 1.

Calculator Assumed

12. [8 marks: 3, 5] [TISC]

In ΔPQR drawn below, the point T is the midpoint of QR. Let **PT** = $\boldsymbol{a}$ and **TR** = $\boldsymbol{b}$.



(a) Find **PR** and **PQ** in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

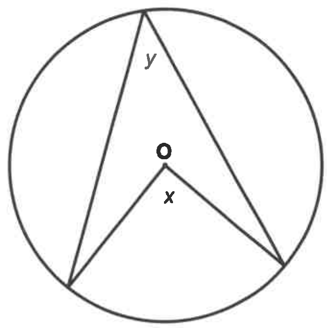
(b) If T is equidistant to P and R, use a vector method to prove that $\angle$QPR = 90°.

16 Geometric Proofs and Circle Properties

Calculator Assumed

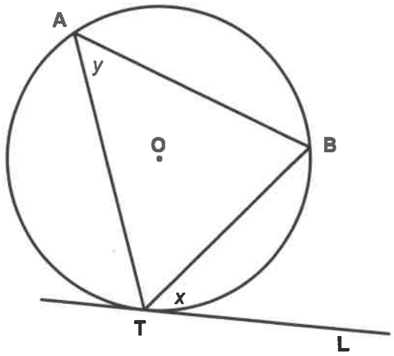
1. [4 marks]

In the accompanying diagram, O is the centre of the circle. Prove that $x = 2y$.



2. [4 marks]

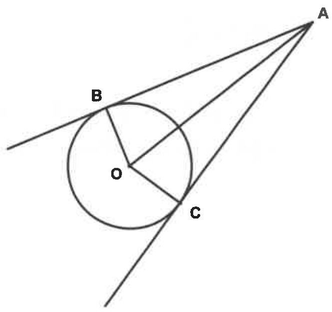
In the accompanying diagram, TL is a tangent to a circle with centre O at T. $\angle BTL = x$ and $\angle TAB = y$. Prove that $x = y$.



Calculator Assumed

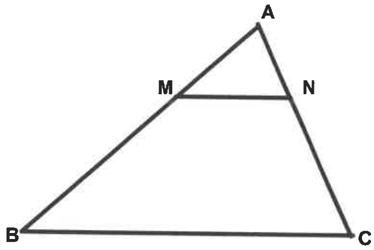
3. [3 marks]

AB and AC are tangents to the circle with centre O.
Prove that ΔABO and ΔACO are congruent.



4. [7 marks: 3, 2, 2]

In ΔABC, the points M and N divide the sides AB and AC respectively in the ratio 1 : 3.



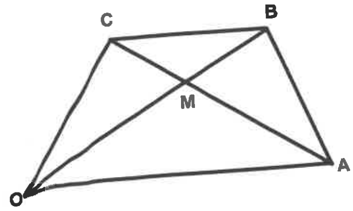
(a) Prove that ΔAMN and ΔABC are similar.

(b) Hence, deduce that BC = 4MN.

(c) Prove that MN is parallel to BC.

Calculator Assumed

5. [5 marks: 3, 2]

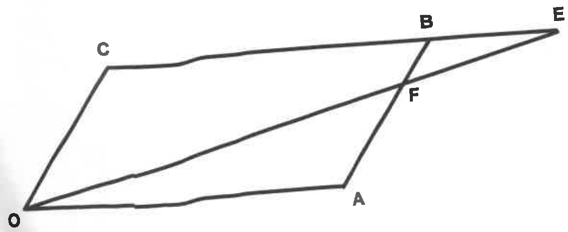


OABC is a trapezium with OA parallel to CB.
The diagonals OB and AC intersect at M such that AM : MC = 3 : 1 .

(a) Prove that ΔMOA and ΔMBC are similar.

(b) Hence deduce that OA = 3BC.

6. [5 marks: 3, 2]



OABC is a parallelogram with OA parallel and congruent to CB. The point F divides AB in the ratio 2 : 1. OF extended meets the CB extended at E.

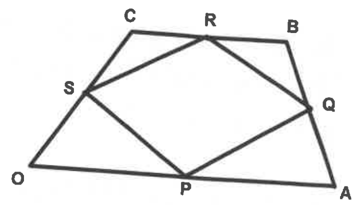
(a) Prove that ΔFOA and ΔFEB are similar.

(b) Hence, deduce that F divides OE in the ratio 2 : 1.

Calculator Assumed

7. [5 marks]

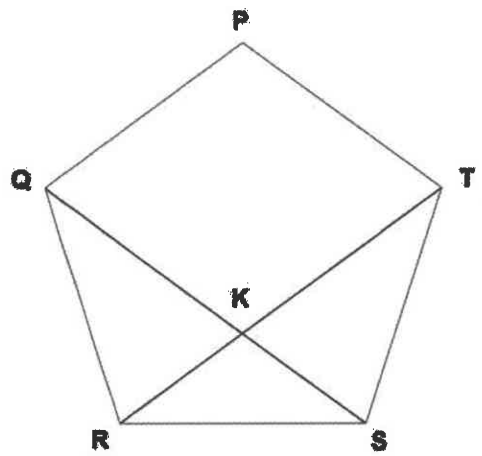
OABC is a trapezium with OA parallel to CB. P, Q, R and S are respectively the midpoints of OA, AB, BC and OC. Prove that the midpoints of the sides of a trapezium form a parallelogram, that is PQRS is a parallelogram.



Calculator Assumed

8. [9 marks: 2, 2, 5]

PQRST is a *regular* pentagon of side length 10 cm.



(a) Find the size of ∠ RST. Justify your answer.

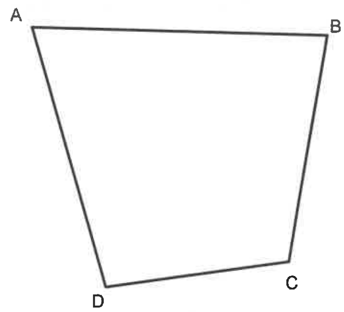
(b) Prove that ∠ STR = 36°.

(c) Find the length of KT. Show clearly how you obtained your answer.

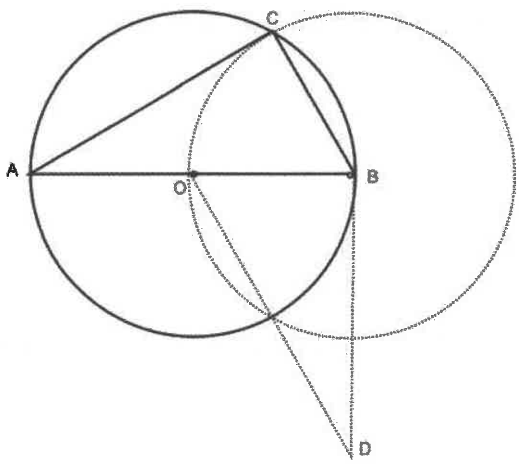
Calculator Assumed

9. [7 marks: 4, 3] [TISC]

(a) ABCD is a quadrilateral.
$\angle DAB + \angle DCB = 180^\circ$ and
$\angle ADC + \angle ABC = 180^\circ$.
Prove that there is a circle that passes through A, B, C and D.



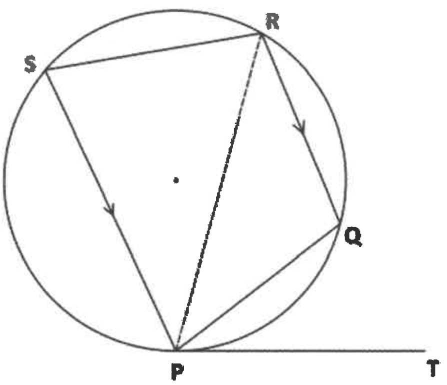
(b) AB is the diameter of the circle with centre O. B is the centre of another circle passing through O. The two circles intersect at C. BD is a tangent to the circle with centre O. If AC = BD, prove that $\angle BOD = \angle CBA$.



Calculator Assumed

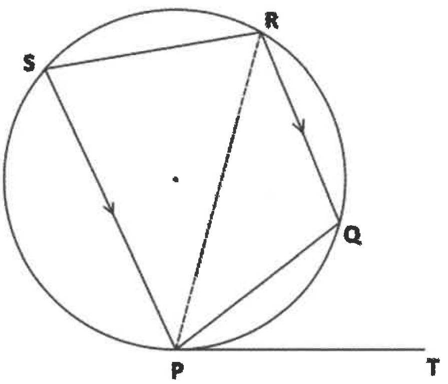
10. [7 marks: 3, 4] [TISC]

PQRS is a cyclic quadrilateral with RQ parallel to SP. PT is a tangent to the circle. The line PR bisects ∠SPQ.



(a) Prove that PQ bisects ∠TPR.

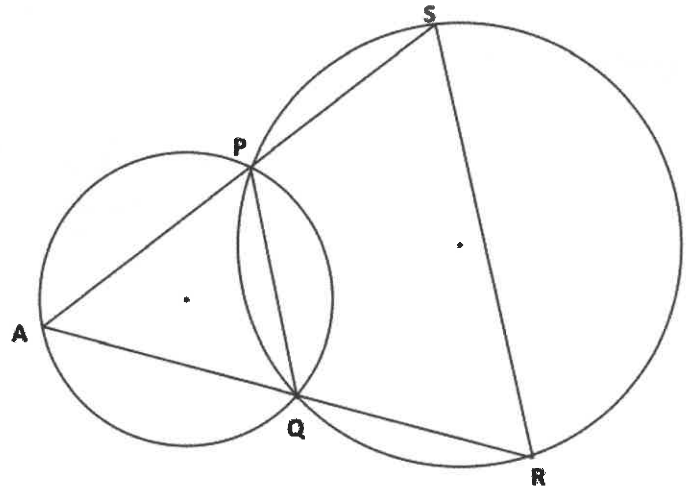
(b) Prove that PQ = SR.
[Hint: Prove that ΔRPQ is congruent to ΔQSR.]



Calculator Assumed

11. [7 marks: 3, 4]

In the diagram below, the two circles intersect at P and Q. S and P are points on the circumference of the larger circle. The points A, P and S are collinear. The points A, Q and R are collinear.



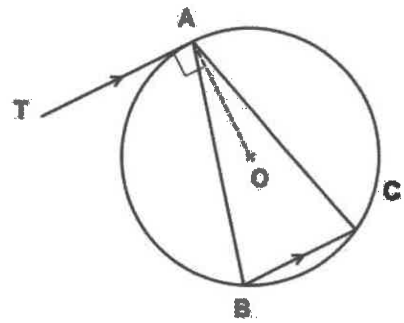
(a) Prove that ΔAPQ and ΔARS are similar.

(b) Given that AQ = 10 cm, AP = QR = 8 cm, find PS.

Calculator Assumed

12. [3 marks]

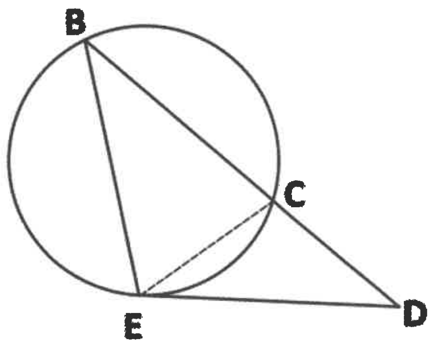
The points A, B and C lie on a circle centre O. TA is the tangent to the circle at A. If TA is parallel to BC, prove that ΔABC is isosceles.



13. [5 marks]

The points B, C and E lie on the same circle. The chord BC extended meets the tangent to the circle at E at the point D.

(a) Prove that $ED^2 = CD \times BD$.

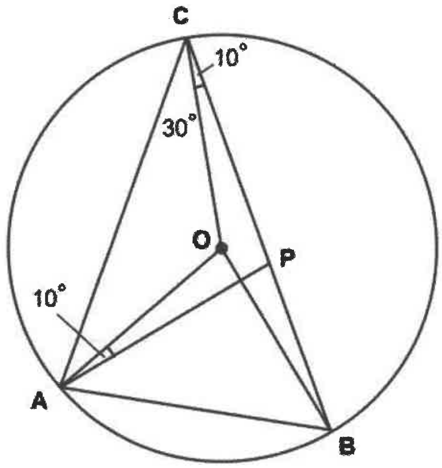


(b) BD is 20 cm long and the point C divides BD in the ratio 3: 2. Hence, find the length of DE.

Calculator Assumed

14. [6 marks: 4, 2]

(a) The points A, B and C lie on a circle centre O. ∠ACO = 30° and ∠BCO = 10°. P is a point on the chord BC such that ∠OAP = 10°. Find with reasons ∠APB.



(b) If the points A, B and C lie on the circumference of a circle and O is a point inside the circle, prove or disprove the conjecture that if ∠AOB = 2∠ACB, then O must be the centre of the circle.

17 Trigonometric Equations I (Simple trigonometric ratios)

Calculator Free

1. [0 marks]

   Complete the following table. Give answers with rational denominators.

Angle θ in degrees	Angle θ in radians	sin θ	cos θ	tan θ
0°				
30°				
45°				
60°				
90°				
120°				
135°				
150°				
180°				
210°				
225°				
240°				
270°				
300°				
315°				
330°				
360°				

Calculator Free

2. [7 marks: 1 each]

Complete the table below. Leave all answers with rational denominators.

Trigonometric Ratio	Value
$\text{cosec } 60^\circ$	
$\sec 150^\circ$	
$\cot 300^\circ$	
$\text{cosec } (-135^\circ)$	
$\cot \frac{7\pi}{6}$	
$\sec \frac{4\pi}{3}$	
$\text{cosec} \left(-\frac{\pi}{4}\right)$	

---

3. [6 marks: 2, 2, 2]

Solve for θ within the given domain:

(a) $\sin\theta = \frac{\sqrt{3}}{2}$ where $0^\circ \le \theta \le 360^\circ$

(b) $\cos\theta = -\frac{\sqrt{2}}{2}$ where $0 \le \theta \le 2\pi$

(c) $\tan\theta = \sqrt{3}$ where $-\pi < \theta \le \pi$

Calculator Free

4. [13 marks: 3, 3, 4, 3]

Given that $\cos 15^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$, solve for θ in:

(a) $\cos(\theta + 5^\circ) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ for $0 \le \theta \le 360^\circ$

(b) $\cos\theta = -\left(\dfrac{\sqrt{6}+\sqrt{2}}{4}\right)$ for $0 \le \theta \le 360^\circ$

(c) $\sin\theta = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ for $0 \le \theta \le 360^\circ$

(d) $\sec\theta = \sqrt{6} - \sqrt{2}$ for $0 \le \theta \le 360^\circ$

Calculator Free

5. [11 marks: 1, 3, 4, 3]

Given that $\sin\left(\frac{\pi}{10}\right) = \frac{\sqrt{5}-1}{4}$, find all solutions (in radians) to:

(a) $\sin x = \frac{\sqrt{5}-1}{4}$

(b) $\sin x = \frac{1-\sqrt{5}}{4}$

(c) $\cos x = \frac{\sqrt{5}-1}{4}$

(d) $\operatorname{cosec} x = 1+\sqrt{5}$

Calculator Free

6. [13 marks: 3, 3, 4, 3]

Solve for all values of θ (in degrees):

(a) $\sin\theta = \cos\theta$

(b) $(\cos\theta - 2)(2\cos\theta + 1) = 0$

(c) $2\sin^2\theta + 3\sin\theta - 2 = 0$

(d) $\sec^2\theta - 4\sec\theta + 4 = 0$

18 Trigonometric Identities I (Pythagorean)

Calculator Free

1. [19 marks: 3, 3, 3, 3, 4, 3]

Simplify each of the following expressions:

(a) $\dfrac{1-\cos^2 A}{\tan^2 A}$

(b) $\dfrac{1}{\sin x \tan x + \cos x}$

(c) $\dfrac{\sin B}{1-\cos B} - \dfrac{1}{\tan B}$

(d) $\dfrac{\cos Q}{1+\sin Q} + \dfrac{\cos Q}{1-\sin Q}$

Calculator Free

1. (e) $\dfrac{1+\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{1+\sin\theta}$

(f) $\dfrac{\sin^3 x-\cos^3 x}{1+\sin x\cos x}$ [Hint: $(a^3-b^3)=(a-b)(a^2+ab+b^2)$]

---

2. [4 marks: 2, 2]

Prove each of the following identities:

(a) $(\sin x+\cos x)^2 = 1 + 2\sin x\cos x$

(b) $\dfrac{1-\sin^2 B}{1-\cos^2 B} = \cot^2 B$

Calculator Free

3. [11 marks: 3, 4, 4]

(a) Prove $\dfrac{1}{1+\sin\theta}+\dfrac{1}{1-\sin\theta}=2\sec^2\theta$.

(b) Prove $(1+\tan^2 P)(1-\sin^2 P)=1$

(c) Prove $\dfrac{1-2\cos^2 x}{\sin x+\cos x}=\sin x-\cos x$

Calculator Free

4. [10 marks: 4, 3, 3]

(a) Prove $\dfrac{1+\sin M}{\cos M} = \dfrac{\cos M}{1-\sin M}$

(b) Prove $\sec x \operatorname{cosec} x = \tan x + \cot x$

(c) Prove $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$

Calculator Free

5. [8 marks: 2, 2, 4]

(a) Prove $\dfrac{1}{1+\cot x}=\dfrac{\tan x}{1+\tan x}$

(b) Prove $\dfrac{\sin x}{1+\cos x}=\dfrac{1}{\operatorname{cosec} x+\cot x}$

(c) Prove $\dfrac{\cos x}{1+\sin x}=\sec x-\tan x$

Calculator Free

6. [7 marks: 4, 3]

Prove each of the following:

(a) $\dfrac{\text{cosec}\, x+1}{\text{cosec}\, x-1} = \tan^2 x + 2\tan x \sec x + \sec^2 x$

(b) $\dfrac{1}{\sec^2 x-1} = \text{cosec}^2\, x - 1$

19 Trigonometric Identities II (Add/Sub Formulae)

Calculator Free

1. [10 marks: 3, 3, 4]

Use an appropriate trigonometric identity to find the exact value of :

(a) $\sin 75^\circ$

(b) $\cos 165^\circ$

(c) $\tan \dfrac{7\pi}{12}$

Calculator Free

2. [8 marks: 1, 1, 3, 3]

Given that $\sin A = \frac{4}{5}$ and $0 < A < \frac{\pi}{2}$, find the exact value of:

(a) $\cos A$

(b) $\tan A$

(c) $\sin(\frac{\pi}{2} + A)$

(d) $\cos(\frac{\pi}{4} - A)$

Calculator Assumed

3. [10 marks: 1, 1, 2, 2, 4]

Given that $\sin A = \frac{3}{5}$ and $\cos B = \frac{1}{4}$, where A and B are acute, use appropriate trigonometric identities (relationships) to find the exact value of:

(a) cos A

(b) sin B

(c) sin(A + B)

(d) cos (A – B)

(e) tan (A + B)

Calculator Assumed

4. [9 marks: 1, 1, 2, 2, 3]

Given that $\sin P = \frac{5}{13}$ and $\cos Q = -\frac{15}{17}$, where $\frac{\pi}{2} \leq P \leq \pi$ and $\frac{\pi}{2} \leq Q \leq \pi$, use appropriate trigonometric identities to find the exact value of:

(a) cos P

(b) sin Q

(c) sin(P – Q)

(d) cos (P + Q)

(e) tan (P – Q)

Calculator Assumed

5. [10 marks: 2, 3, 5]

(a) Prove that $\sin(-A) = -\sin A$

(b) Prove that $\dfrac{\sin(A - B)}{\sin A \sin B} = \cot B - \cot A$

(c) Use your answers in parts (a) and (b) to rewrite $\dfrac{\sin 2x}{\sin x \sin 3x} + \dfrac{\sin 2x}{\sin 3x \sin 5x}$

in the form $\dfrac{\sin a}{\sin b \sin c}$, giving the values of a, b and c.

20 Trigonometric Identities III (Double angle)

Calculator Free

1. [11 marks: 1, 1, 3, 3, 3]

Given that $\sin P = \frac{1}{4}$ and $\cos Q = \frac{2}{3}$, where $\frac{\pi}{2} \le P \le \pi$ and $\frac{3\pi}{2} \le Q \le 2\pi$, find the exact value of:

(a) $\cos P$

(b) $\sin Q$

(c) $\cos (P + Q)$

(d) $\tan 2Q$

(e) $\sin \frac{Q}{2}$

Calculator Free

2. [15 marks: 3, 3, 4, 5]

Prove each of the following identities:

(a) $\cos^4 x - \sin^4 x = \cos 2x$

(b) $\dfrac{\sin 2t}{1+\cos 2t} = \tan t$

(c) $\cos 6x = 4\cos^3 2x - 3\cos 2x$

(d) $\dfrac{1-\sin 2t}{\cos 2t} = \dfrac{1-\tan t}{1+\tan t}$

Calculator Free

3. [11 marks: 3, 4, 4]

   Prove each of the following:

   (a) $\dfrac{\cos x - \sin 2x}{\cos 2x + \sin x - 1} = \cot x$

   (b) $\cot \theta + \tan \theta = 2 \operatorname{cosec} 2\theta$

   (c) $\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}$

Calculator Free

4. [9 marks: 3, 3, 3]

(a) Prove that $\sqrt{1-\cos x} = \sqrt{2}\sin\dfrac{x}{2}$.

(b) Prove that $\dfrac{\sin x}{1+\cos x} = \tan\dfrac{x}{2}$.

(c) Prove that $\tan^2\left(\dfrac{3x}{2}\right) = \dfrac{1-\cos 3x}{1+\cos 3x}$

Calculator Free

5. [13 marks: 4, 3, 6]

(a) Prove that $(a\cos x + b\sin x)^2 + (b\cos x - a\sin x)^2 = a^2 + b^2$.

(b) Prove that $\cos^4 2x - \sin^4 2x = \cos 4x$

(c) Prove that $\dfrac{1}{1+\tan x} - \dfrac{1}{1-\tan x} = -\tan 2x$.

Calculator Free

6. [11 marks: 3, 3, 2, 3]

(a) Prove that $\cos 3A = 4\cos^3 A - 3\cos A$.

(b) Prove that $\sin 3A = 3\sin A - 4\sin^3 A$

(c) Given that $\sin\theta = \dfrac{1}{4}$, where $0 < \theta < \dfrac{\pi}{2}$, find:

(i) $\sin 3\theta$

(ii) $\cos 3\theta$

21 Trigonometric Identities IV (Product to Sum and Sum to Product)

Calculator Free

1. [8 marks: 3, 3, 2]

(a) Use an appropriate compound angle formula to prove that

$$\sin\left(\frac{A+B}{2}\right) + \sin\left(\frac{A-B}{2}\right) = 2\sin\frac{A}{2}\cos\frac{B}{2}.$$

(b) Use the result in (a) to prove that $\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$.

(c) Use the result in (b) to evaluate $\sin 75^\circ + \sin 15^\circ$

Calculator Free

2. [9 marks: 3, 4, 2]

(a) Use an appropriate compound angle formula to prove that

$$\cos\left(\frac{A+B}{2}\right) - \cos\left(\frac{A-B}{2}\right) = -2\sin\frac{A}{2}\sin\frac{B}{2}.$$

(b) Use the result in (a) to prove that $\cos P - \cos Q = 2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{Q-P}{2}\right)$.

(c) Use the result in (b) to evaluate $\cos 255^\circ - \cos 15^\circ$

Calculator Free

3. [11 marks: 2, 3, 2, 2, 2]

(a) Prove that $\cos P \cos Q = \frac{1}{2}[\cos(P + Q) + \cos(P - Q)]$.

(b) Use the result in (a) to prove that $\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$.

(c) Use the results in (a) and/or (b) to evaluate $\cos^2 15^\circ$.

(d) Use the results in (a) and/or (b) to evaluate $\sin 15^\circ \cos 15^\circ$.

(e) Hence, evaluate $\cot 15^\circ$.

Calculator Free

4. [12 marks: 3, 4, 5]

(a) Prove that $\dfrac{\sin 4A + \sin 2A}{\sin 4A - \sin 2A} = \tan 3A \cot A$.

(b) Prove that $\sin 5\theta + 2\sin 3\theta + \sin\theta = 4\sin 3\theta\cos^2\theta$.

(c) Prove that $\dfrac{\sin P + \cos(2Q - P)}{\cos P - \sin(2Q - P)} = \cot\left(\dfrac{\pi}{4} - Q\right)$

Calculator Free

5. [7 marks: 1, 3, 3]

(a) Prove that $\sin A \cos A = \frac{1}{2} \sin 2A$

(b) Prove that $\sin 40^\circ \cos 40^\circ \cos 80^\circ = \frac{\sin 20^\circ}{4}$.

(c) Use your answer in (b) to prove that $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$.

22 Trigonometric Identities V (Auxiliary Angles)

Calculator Assumed

1. [8 marks: 4, 4]

(a) Given that $\cos x + \sqrt{3}\sin x = \text{R}\sin(x + \alpha)$ where $\text{R} > 0$ and $0 \le \alpha \le \dfrac{\pi}{2}$, use the formula for sin (A + B) to find in exact form the value of R and the exact value of α.

(b) Hence, find the maximum value (in exact form) for $y = \sqrt{3}\cos x + 3\sin x$ and the smallest positive value of x at which this occurs.

Calculator Assumed

2. [10 marks: 4, 3, 3]

Compare $4 \sin x + 7 \cos x$ with the expansion of $R \sin(x + \alpha)$ where $0 \leq \alpha \leq 90°$.
Hence, find the exact value of R and the value of α to 2 decimal places

Hence, find:

(a) the maximum value (in exact form) for the expression $8 \sin x + 14 \cos x$ and the smallest positive value of x at which this occurs.

(b) the maximum value (in exact form) for the expression $-4 \sin x - 7 \cos x$ and the smallest positive value of x at which this occurs.

Calculator Assumed

3. [10 marks: 4, 3, 3]

Compare 5 cos x + 8 sin x with the expansion of form R cos(x – α) where $0 \le \alpha \le \frac{\pi}{2}$. Hence, find the exact value of R and α to 4 decimal places.

Hence, find:

(a) the maximum value (in exact form) for the expression cos x + 1.6 sin x and the values of x where $0 \le x \le 2\pi$ at which this occurs.

(b) the minimum value (in exact form) for the expression 10 sin x + 16 cos x and the values of x where $0 \le x \le 2\pi$ at which this occurs.

Calculator Assumed

4. [8 marks]

Use an appropriate trigonometric method to find the minimum value (in exact form) for $f(\theta) = 10 + 3\sin\theta + 5\cos\theta$ where $0 \le \theta \le 360^\circ$. Give also the smallest positive value for θ at which the minimum value of $f(\theta)$ occurs.

23 Trigonometric Equations II

Calculator Free

1. [9 marks: 5, 4]

Solve for x within the given domain:

(a) $2\cos^2 x + 3\sin x = 0$ for $0 \le x \le 360^\circ$

(b) $\cos x - 3\sec x - 2 = 0$ for $0 \le x \le 2\pi$

Calculator Free

2. [13 marks: 3, 4, 6]

(a) Solve for x in $\cos x + \sqrt{3}\sin x = 0$ where $0 \le x \le 360^\circ$:

(b) Solve for x in $\sin x - \cos 2x = 0$ where $-\pi < x \le \pi$.

(c) Find all values of x (in degrees) in $\cos x + \sin 2x = 0$.

Calculator Free

3. [9 marks: 4, 5]

(a) Solve for θ in $\cos 2\theta + \cos\theta + 1 = 0$ for $0 \le \theta \le 2\pi$.

(b) Find all solutions (in radians) to $\sin 2\theta - \sin\theta = 0$.

Calculator Free

4. [9 marks: 4, 5]

(a) Find all solutions (in radians) for θ in $3\tan^2\theta + 5\sec\theta + 1 = 0$.

(b) Find all solutions (in degrees) for θ in $\tan\theta + \cot\theta - 2\sec\theta = 0$.

Calculator Free

5. [9 marks: 4, 5]

(a) Solve for all values of θ in $\tan\left(2\theta + \frac{\pi}{6}\right) = -1$.

(b) Solve for all values of θ in $2\cos 2\theta + 2\sin^2\theta - 9\cos\theta - 5 = 0$.

Calculator Free

6. [12 marks: 3, 5, 4]

(a) Solve for all values of θ in $\sqrt{3}\sin\theta + \cos\theta = 0$

(b) Solve for all values of θ in $\sqrt{3}\sin\theta + \cos\theta = 1$

(c) Solve for all values of θ in $\sqrt{3}\sin\theta + \cos 2\theta = 1$.

Calculator Free

7. [11 marks: 5, 6]

(a) Solve for all values of θ in $\cos\theta + \sin\theta = 1$

(b) Solve for all values of θ in $1 + \sqrt{3}\tan\theta = \sqrt{3}\sec\theta$.

Calculator Free

8. [9 marks: 4, 5]

(a) Solve for all values of θ in $\cos\theta + \cos 3\theta = 0$.

(b) Solve $\cos\theta + \cos 3\theta + \cos 7\theta = 0$ for $0 \leq \theta \leq 180^{\circ}$.

Calculator Assumed

9. [10 marks: 1, 3, 6]

(a) Show that $\sin 2\theta + \sin 3\theta \equiv 2\sin\dfrac{5\theta}{2}\cos\dfrac{\theta}{2}$.

(b) Use your result in (a) to solve for all values of θ in $\sin 2\theta + \sin 3\theta = 0$

(c) Use your result in (a) to solve for all values of θ in $\sin 2\theta + \sin 3\theta - \sin\theta = 0$.

Calculator Assumed

10. [11 marks: 4, 2, 5]

(a) Use the formula for tan 2A to show that $\tan\frac{\pi}{8} = -1 + \sqrt{2}$.

(b) Use your answer in (a) to find all solutions to $\sqrt{2}\cos\theta - \cos\theta - \sin\theta = 0$.

(c) Given that $\sin\frac{3\pi}{8} = \frac{1}{\sqrt{4-2\sqrt{2}}}$ and using the answer in (a),

solve for θ in $\sin\theta - (\sqrt{2}-1)\cos\theta = 1$ for $0 < \theta \leq 2\pi$.

24 Trigonometric Graphs

Calculator Free

1. [6 marks]

Complete the following table.

Function	Period	Amplitude	Phase Shift
$y = 2\sin(2x^\circ)$			
$y = -4\cos(\frac{x}{2} + 30^\circ)$			
$v = 10\tan(3t + \pi)$			
$Q = 5\sin(\frac{\pi}{2} - t)$			
$y = \frac{\sqrt{2}}{2}\cos(\pi t) + 100$			
$T = 5 - \sin(\frac{\pi}{4} - \theta)$			

2. [5 marks]

Complete the table below.

Function	Minimum value of function	Maximum value of function
$y = 3\sin t$		
$y = 20\cos(\frac{2x}{3} - 45^\circ)$		
$v = 5\tan\theta$		
$M = 2\sin(\frac{\pi}{2} - 3t) + 4$		
$y = 5 - \cos(2\pi t)$		

Calculator Free

3. [6 marks: 3, 3]

A trigonometric function has equation $y = -4\sin(2x + 30^\circ)$ for $0^\circ \le x \le 360^\circ$. Use an algebraic method to find:

(a) the maximum value for y and the corresponding value(s) for x.

(b) the minimum value for y and the corresponding value(s) for x.

---

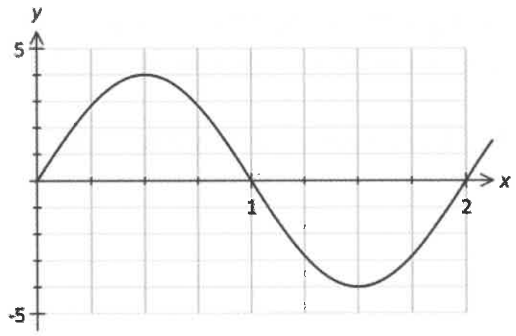
4. [5 marks]

A trigonometric function has equation $P = a\cos(bt + \frac{\pi}{4})$. Find the values of a (where $a > 0$) and b given that P has a maximum value of 4 when $t = \frac{\pi}{4}$.

Calculator Free

5. [3 marks]

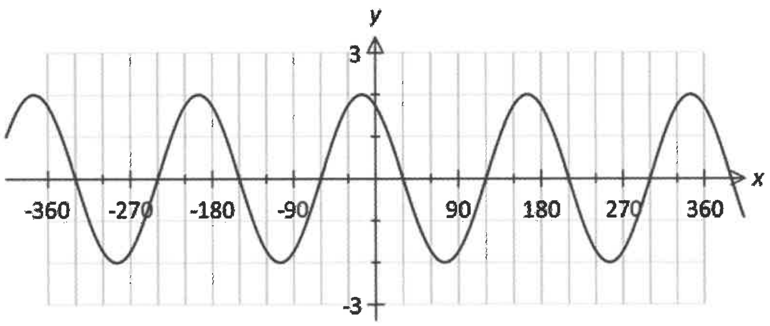
The accompanying diagram shows the graph of a trigonometric function. State the amplitude and period of the function. Hence, give the equation of this function.



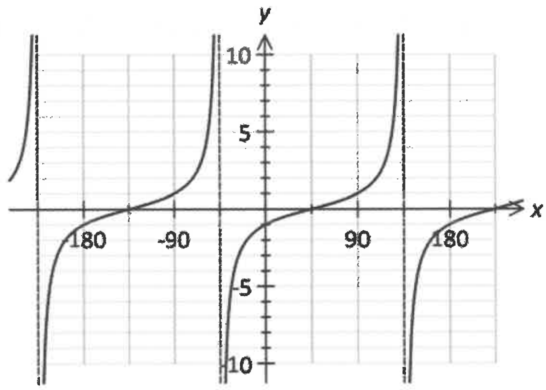
6. [7 marks: 4, 3]

Find the equation of the following trigonometric functions:

(a)



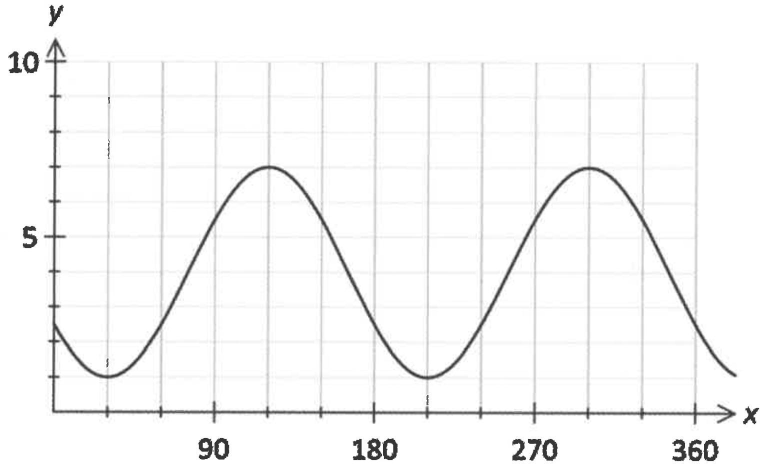
(b)



Calculator Free

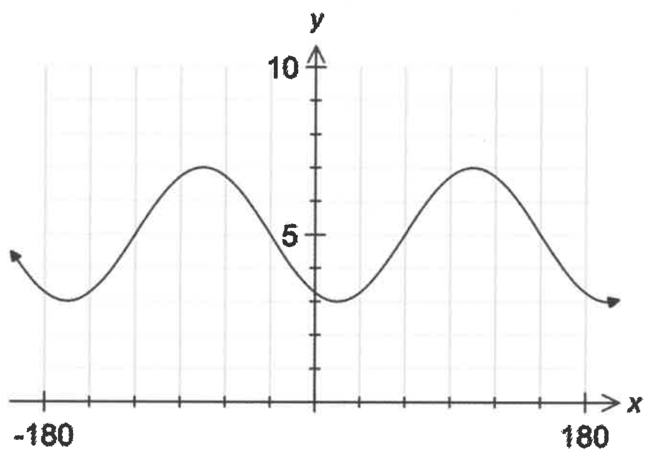
7. [4 marks]

The graph of $y = a + b\sin(cx + d)$ is shown below.
Determine the values of a, b, c and d.



8. [4 marks]

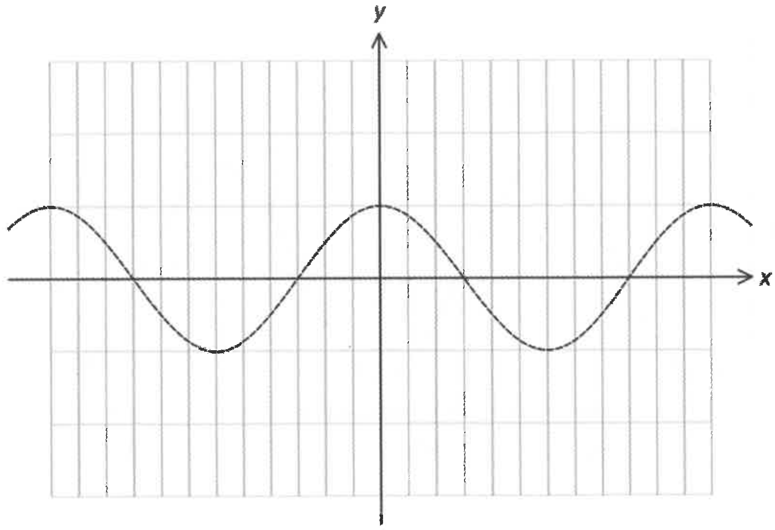
The graph of $y = a + b\cos(cx + d)$ is shown below.
Determine the values of a, b, c and d.



Calculator Free

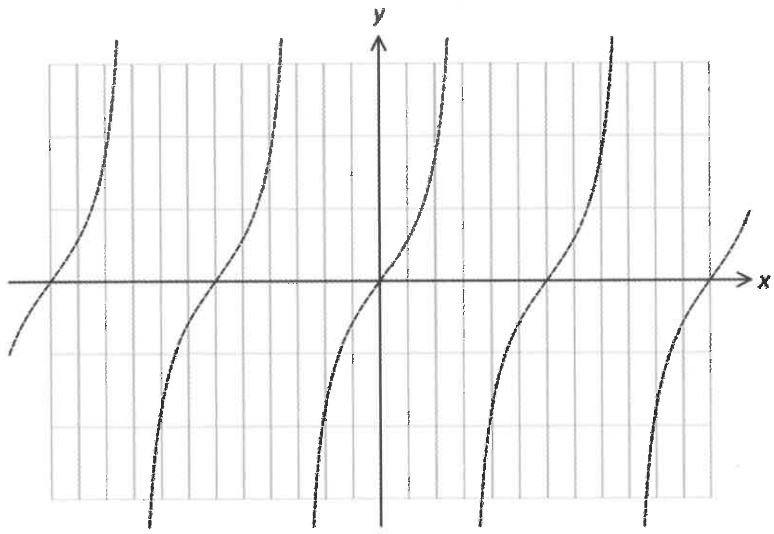
9. [3 marks]

The graph of $y = \cos(ax)$ is shown below.
On the same diagram, sketch the graph of $y = \sec(ax)$.



10. [3 marks]

The graph of $y = \tan(ax)$ is shown below.
On the same diagram, sketch the graph of $y = \cot(ax)$.



Calculator Assumed

11. [5 marks: 2, 1, 2]

Consider the curve with equation $y = 2\sec\left(\frac{x}{2}\right)$ for $-720^{\circ} < x < 720^{\circ}$.

(a) Determine the coordinates of the maximum turning point(s) of this curve.

(b) Determine the period of this curve.

(c) Determine the equations of the vertical asymptotes.

---

12. [5 marks: 2, 1, 2]

Consider the curve with equation $y = 2 + \operatorname{cosec}(2x + 30^{\circ})$ for $-180^{\circ} < x < 180^{\circ}$.

(a) Determine the coordinates of the minimum turning point(s) of this curve.

(b) Determine the period of this curve.

(c) Determine the equations of the vertical asymptotes.

Calculator Assumed

13. [9 marks: 1, 1, 2, 2, 3]

The body temperature θ (Celsius) of a reptile in summer at time t hours after midnight is given by $\theta = 15 - 5\sin\left(\frac{\pi t}{12}\right)$.

(a) State the period for θ.

(b) What is the range of body temperature experienced by the reptile?

(c) Find the minimum body temperature of the reptile and state when this first occurs after midnight.

(d) Find the maximum body temperature of the reptile and state when this first occurs after midnight.

(e) Find for how many hours in a 24 hour day, the body temperature of the reptile is below 16° Celsius. Give your answer to the nearest minute.

Calculator Assumed

14. [10 marks: 2, 3, 5]

The water depth, h metres, measured from the bottom of a harbour, t hours after 6 am is modelled by the equation $h = 12 - 4\cos\left(\frac{\pi t}{6} - \frac{\pi}{4}\right)$ metres.

(a) Determine when the water depth is at its lowest in a 24-hour day (from 6 am). State the lowest depth.

(b) Repairs to the harbour can only be undertaken if the water depth is below 10 metres. What times of the day can this occur?

(c) The water depth is above k metres for 20% of a 24-hour day. Find k.

25 Matrix Algebra

Calculator Free

1. [6 marks: 1, 1, 4] [TISC]

Consider the matrices $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) Given that matrix **X** can be added with matrix **A**, what is the dimension (size) of matrix **X** ?

(b) Given that **BY** = **YB**, what is the dimension (size) of matrix **Y** ?

(c) Consider the matrices, $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ k \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \end{pmatrix}$.

Two different matrices are selected from the three given and then multiplied together. State all the possible products.

Calculator Free

2. [5 marks: 1, 1, 1, 1, 1] [TISC]

$\mathbf{A} = \begin{pmatrix} -27 & -3 \\ 1 & 5 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 4 & -3 \\ 8 & 5 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} -15 & -12 \\ 25 & 20 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} -15 & 12 \\ 25 & 20 \end{pmatrix}$

$\mathbf{E} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$ $\mathbf{F} = \begin{pmatrix} -27 \\ 1 \end{pmatrix}$ $\mathbf{G} = \begin{pmatrix} 5 & 4 \end{pmatrix}$ $\mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Choose a matrix from the list above to make the following statements true.
Write the name of the matrix in the space provided.

(a) .... is a square singular matrix.

(b) $\mathbf{E} \times \mathbf{G} =$ ....

(c) $\mathbf{D} \times \mathbf{I} =$ ....

(d) .... $- 3\mathbf{B} = \mathbf{A}$

(e) $\mathbf{B} \times$ ...... $= \mathbf{F}$

---

3. [5 marks: 2, 3] [TISC]

Let $\mathbf{A} = \begin{pmatrix} k & 1 \\ 8 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ -8 & k \end{pmatrix}$

(a) Find the value(s) of k if $\mathbf{A}$ is a non-singular matrix.

(b) Find the value(s) of k if $\mathbf{A} \times \mathbf{B} = \mathbf{A} + \mathbf{B}$.

Calculator Assumed

4. [5 marks: 1, 2, 2] [TISC]

Given that $\mathbf{A} = \begin{pmatrix} k-1 & 0 \\ 0 & k+1 \end{pmatrix}$.

(a) Find in terms of k, the determinant of matrix $\mathbf{A}$.

(b) Find the value(s) of k for which $\mathbf{A}$ is singular.

(c) Given that A is non-singular, find $\mathbf{A}^{-1}$ in terms of k.

---

5. [6 marks: 2, 2, 2] [TISC]

Given that $\mathbf{A} = \begin{pmatrix} a & 1 \\ b & a \end{pmatrix}$.

(a) Find the relationship between a and b such that $\mathbf{A}$ is a singular matrix.

(b) Given that $b = 4$, find the value(s) of a for which $\mathbf{A}$ is non-singular.

(c) Explain clearly why $\mathbf{A}$ will always have an inverse if $b < 0$.

Calculator Free

6. [6 marks: 1, 2, 3]

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -1 & 5 \\ 5 & -2 \end{pmatrix}$.

(a) Find $\mathbf{B}^{-1}$.

(b) Find $\mathbf{C} \times \mathbf{B}^{-1}$.

(c) Find $\mathbf{X}$ if $(\mathbf{A} + \mathbf{X})\,\mathbf{B} = \mathbf{C}$.

Calculator Free

7. [4 marks: 2, 2] [TISC]

(a) If **A** is a non-singular square matrix, show that if $\mathbf{A}^2 = \mathbf{A}$, then $\mathbf{A} = \mathbf{I}$ where **I** is the appropriate identity matrix.

(b) Find a 2×2 non-zero matrix **A**, where $\mathbf{A}^2 = \mathbf{A}$ and $\mathbf{A} \neq \mathbf{I}$. (**I** is the 2×2 identity matrix.)

---

8. [6 marks: 3, 3]

(a) Given that **P** and **Q** are square matrices and $\mathbf{PQ} = \mathbf{P} + \mathbf{Q}$, show that $\mathbf{P} = \mathbf{Q}(\mathbf{Q} - \mathbf{I})^{-1}$, where **I** is the appropriate identity matrix.

(b) Given that **A** and **B** are 2×2 non-zero diagonal matrices, prove that **A** and **B** are commutative under multiplication.

Calculator Free

9. [9 marks: 2, 3, 4]

Given that the non-singular matrix **A**, where $\mathbf{A}\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}$ and $\mathbf{A}\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}3\\2\\-1\end{pmatrix}$

find each of the following. Justify each of your answers.

(a) $\mathbf{A}\begin{pmatrix}2\\-1\\3\end{pmatrix}$

(b) $\mathbf{A}^2\begin{pmatrix}1\\-1\\1\end{pmatrix}$

(c) $\mathbf{A}^{-1}\begin{pmatrix}1&3&4\\0&2&2\\2&-1&1\end{pmatrix}$

Calculator Assumed

10. [5 marks]

Given that $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with all non-zero elements, where $|\mathbf{M}| = 1$ and $\mathbf{M}^{-1} = \mathbf{M}^2$, prove that $a + d = -1$.

11. [4 marks]

Let $\mathbf{X}$ be a $n \times 1$ matrix, $\mathbf{A}$ be a $n \times n$ matrix and λ be a real non-zero constant. Given that $\mathbf{A}\,\mathbf{X} = \lambda\mathbf{X}$, prove that $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

Calculator Assumed

12. [7 marks: 2, 2, 3] [TISC]

Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}$.

(a) Show that $\mathbf{A}^2 = \mathbf{A} + k\mathbf{I}$ where k is a real constant.

(b) Use your result in (a) to find $\mathbf{A}^{-1}$ in the form $\alpha\mathbf{A} + \beta\mathbf{I}$.

(c) Find $\mathbf{A}^4$ in terms of $\mathbf{A}$ and $\mathbf{I}$.

Calculator Assumed

13. [8 marks: 3, 1, 4]

(a) Given that **A**, and **B** are non-singular square matrices prove that

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

(b) Hence, or otherwise, prove that $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$.

(c) Given that $\mathbf{A}^2 = 2\mathbf{A} + \mathbf{I}$, find $(\mathbf{A}^2)^{-1}$ in the form $p\mathbf{A} + q\mathbf{I}$ where p and q are real constants and $\mathbf{I}$ is the identity matrix.

26 Systems of Equations

Calculator Free

1. [6 marks: 1, 1, 2, 2]

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$.

(a) Find $\mathbf{A}^{-1}$.

(b) Find the product $\mathbf{A}^{-1}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

(c) Consider the system of equations:

$$x + 2y = 2$$
$$-x + 3y = -2$$

(i) Rewrite the given system of equations in the form **AX** = **B** where **X** is a column matrix and **A** and **B** are appropriate matrices.

(ii) Use a matrix method to solve for x and y.

Calculator Assumed

2. [6 marks: 1, 1, 2, 2]

Let $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$.

(a) Find $\mathbf{A}^{-1}$.

(b) Find the product $\begin{pmatrix} 3 & 4 \end{pmatrix} \times \mathbf{A}^{-1}$.

(c) Consider the system of equations:

$$3x + 5y = 3$$
$$x + 2y = 2$$

(i) Rewrite the given system of equations in the form **XA** = **B** where **X** is a row matrix and **A** and **B** are appropriate matrices.

(ii) Use a matrix method to solve for x and y.

Calculator Free

3. [4 marks]

Given that $\mathbf{A} \times \mathbf{B} = 4\mathbf{I}$, $\mathbf{B} \times \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{A} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$,

where $\mathbf{I}$ is the 2×2 unit matrix, find x and y.

---

4. [5 marks: 1, 4]

Let $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$.

(a) Find product $\mathbf{A} \times \mathbf{B}$.

(b) Show how your answer in (a) can be used to solve for x and y in:

$$3x + 5y = 1 \quad \text{and} \quad x + 2y = 2.$$

27 Applications using Matrices

Calculator Assumed

1. [7 marks: 2, 3, 2]

A budget airline flies from Perth to Sydney and charges two different fares for its passengers; deluxe economy and economy. On a certain flight, there were: 200 fare paying passengers and one and a half times as many economy passengers as deluxe economy passengers
Let d: number of deluxe economy passengers on this flight
e: number of economy passengers on this flight

(a) Use the information given above to write down two equations involving d and e.

(b) Use a method involving the inverse of a matrix to find d and e.

(c) A deluxe economy ticket costs \$349, while an economy ticket costs 30% less. Use a matrix method to find the total amount in fares collected from this flight. You need to show clearly the matrices used, and the operation(s) used on these matrices.

Calculator Assumed

2. [5 marks: 1, 1, 3]

The table below shows the number of hours Jack worked last week at a fast food outlet.

Shift	Weekdays	Weekends
Morning (M)	16	4
Afternoon (A)	8	4
Night (N)	8	0

(a) Write a *row* matrix **A** describing the number of hours Jack worked on each shift on weekdays.

(b) Write a *column* matrix **B** describing the number of hours Jack worked on each shift on weekends.

The rates of pay are $15.00 per hour for weekday morning shifts, $12.00 per hour for weekday afternoon shifts and $20 per hour for weekday night shifts. Jack is paid twice as much per hour for weekend shifts as weekday shifts.

(c) Use matrices **A** and **B** and other matrices as required to find the total amount of money Jack earned last week.

Calculator Assumed

3. [7 marks: 2, 2, 3]

The number of different tickets available for a charity concert at a concert hall is given in the table below.

	Stalls	Gallery
Adults	500	300
Students/pensioners	150	50

The prices for these tickets are given in the table below.

	Stalls	Gallery
Adults	$150	$90
Students/pensioners	$120	$70

(a) Given that all the tickets were sold, use a matrix method to determine the revenue from:

(i) the adult members of audience

(ii) the stalls tickets.

(b) Given that 310 gallery tickets were sold realising a revenue of $26 900. Use a matrix method to determine how many of the gallery tickets were sold to adults?

Calculator Assumed

4. [8 marks: 2, 3, 3]

In a city there are two companies, A and B, that supply gas to households. The table below shows the percentage of customers in each company that will remain with their original supplier and the percentage of customers that will switch to the competitor within two years.

%		From	
		A	B
To	A	65	15
	B	35	85

Initially, there were 750 000 and 250 000 customers with companies A and B respectively. Assume that the total number of customers remain unchanged.

(a) Use a matrix method to determine the number of customers with company A at the end of two years.

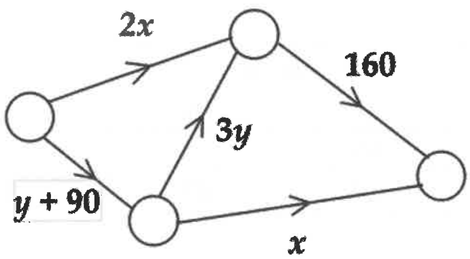
(b) Use a matrix method to determine the number of customers with company A at the end of four years.

(c) Using the table given, will company B ever have 750 000 customers? Justify your answer.

Calculator Assumed

5. [7 marks: 3, 4]

The diagram below shows the flow of fluid (in litres/minute) through a network of pipes. The numbers or letters indicate the flow rate through the pipe concerned. Assume that no fluid is lost in the process.



(a) Write down all equations involving x and y for the given network.

(b) Use a method involving the inverse of a matrix to solve for x and y. Show clearly the matrices involved.

Calculator Assumed

6. [7 marks: 3, 4]

Mr Smart uses the following encryption procedure to code a four digit PIN.

- Replace each digit x, using the function $f(x) = (x + 10)^2$.
  For example, the PIN 5678 becomes $f(5)\ f(6)\ f(7)\ f(8)$, i.e. 225 256 289 324.
- The replacement is then multiplied by $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$. Hence,

  5678 $\Rightarrow$ 225 256 289 324 $\Rightarrow$ 838 805 770 869

(a) Use Mr Smart's procedure to encrypt the PIN 9517. Show clearly each stage of the process.

(b) Mr Smart's partner Ninety9, receives the sequence of numbers 813 621 726 786. Decrypt this sequence of numbers to determine Mr Smart's PIN. Show clearly how you obtained your answer.

Calculator Assumed

7. [8 marks: 3, 3, 2]

The matrix $\mathbf{T} = \begin{array}{cc} & \text{From} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{To} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0.75 & 0.25 & 0.45 & 0.2 \\ 0.15 & 0.45 & 0.25 & 0.1 \\ 0.03 & 0.15 & 0.25 & 0.05 \\ 0.07 & 0.15 & 0.05 & 0.65 \end{pmatrix} \end{array}$ show the percentages of customers from each of the Internet Service Providers A, B, C and D that remain with their original provider and those that will switch providers after 1 year. For example, 75% of customers with A will remain with A after one year while 15% will leave A to join B, 3% will leave A to join C and 7% will leave A to join D.

John starts with B. This may be represented by the column matrix $\mathbf{X} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

(a) Use matrices **T** and **X** to find the probability that John will still be with B:
   (i) after two years.

   (ii) after 10 years

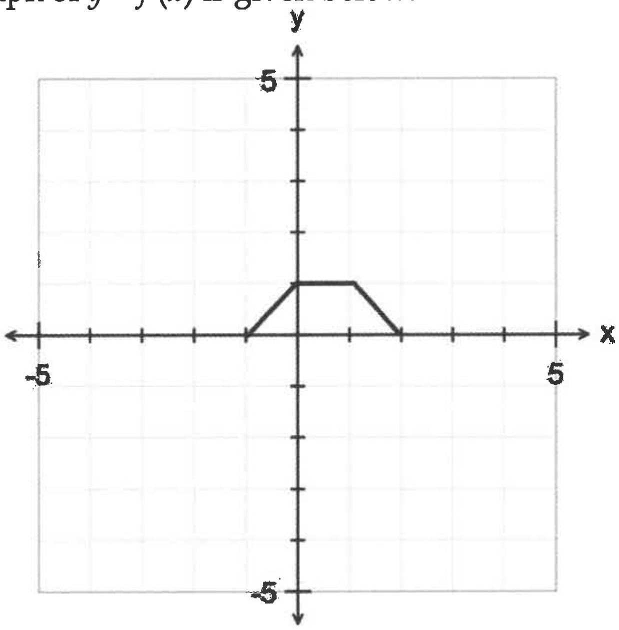
(b) Which will be the dominant provider after 10 years. Justify your answer.

28 Transformation Matrices

Calculator Free

1. [6 marks: 2, 2, 2]

The graph of $y = f(x)$ is given below.



The transformation represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is applied to this curve.

(a) Find the image of the points (–1, 0), (0, 1), (1, 1) and (2, 0).

(b) Sketch on the axes provided above, the graph of the resulting curve.

(c) The equation of the resulting curve is $y = a\,f(bx + c)$. Find a, b and c.

Calculator Free

2. [9 marks: 2, 3, 2, 2] [TISC]

Consider the curve with equation $y = f(x)$. The curve has a maximum point at A $(-1, 3)$ and a minimum point at B $(4, -7)$. The curve $y = f(x)$ is mapped onto the curve $y = g(x)$ by a transformation represented by the matrix $\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(a) Describe the effect the transformation represented by T has on the graph of $y = f(x)$.

(b) Find the coordinates of the images of the points A and B under **T**.

(c) Find the coordinates of the maximum and minimum points on the curve $y = g(x)$.

(d) Find the matrix that maps $y = g(x)$ back to $y = f(x)$.

Calculator Free

3. [6 marks: 1, 1, 2, 2] [TISC]

The transformation T is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(a) Describe in words the transformation T.

(b) The transformation T is applied to the line with equation $y = x$.
Find the equation of the resulting line.

(c) The point A is mapped to the point with coordinates $(k, k + 1)$ under transformation T. Find the coordinates of the point A. Justify your answer.

(d) The transformation T is combined with the transformation represented by matrix **M**. All the entries in matrix **M** are positive. The effects of the combined transformation is represented by the matrix $\begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$.

Find the matrix **M**. Show clearly your reasoning.

Calculator Assumed

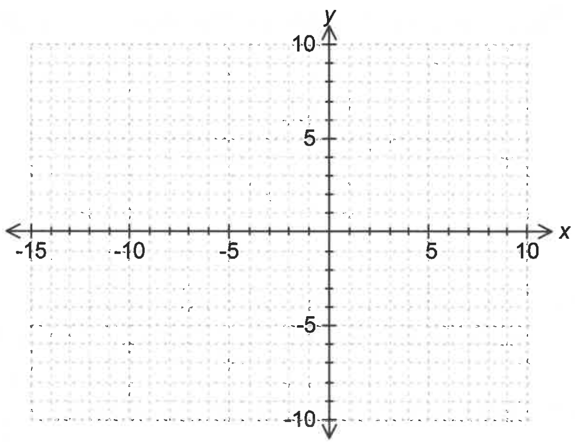
4. [7 marks: 3, 2, 1, 1] [TISC]

Triangle ABC with vertices A (0, 0), B (3, 0) and C (3, 3) is mapped to triangle A´B´C´ by the compound transformation represented by the matrix

$\mathbf{M} = \begin{pmatrix} -2 & -2 \\ 0 & 2 \end{pmatrix}$.

(a) Find the coordinates of the points A´, B´ and C´.

(b) Plot triangle ABC and triangle A´B´C´ on the axes provided below.



(c) The transformation applied is a combination of several simple transformations. What evidence is there to suggest that one of the simple transformations involved is :

(i) an enlargement?

(ii) a reflection ?

Calculator Assumed

5. [6 marks]

Let the matrices $\mathbf{R}_\theta$ and $\mathbf{R}_\phi$ represent the linear transformations of an anticlockwise rotation of θ radians and ϕ radians respectively about the origin. Use these two transformations to prove that:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

and $$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi\,.$$

Calculator Assumed

6. [9 marks: 3, 2, 4] [TISC]

Consider two matrices $\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{T} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

The point A(1, k) is mapped to the point A′′ using **T** followed by **S** as transformation matrices.

(a) Find the coordinates of A′′.

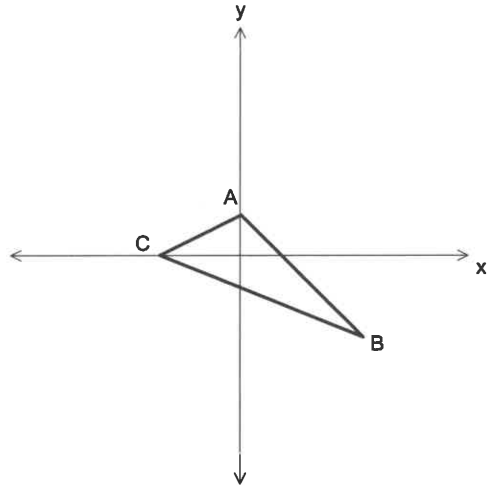
(b) Find a single transformation matrix that will map A′′ back to A. Show how you obtained your answer.

(c) C is a circle of radius 1 with centre at (1, 1). C is transformed to circle C′ by the transformation **T** . Discuss the differences between the original circle C and its image C′. You need to comment on the coordinates of the centre, the radius and area of the two circles.

Calculator Assumed

7. [8 marks: 2, 2, 2, 2]

The transformations **P** and **Q** represented by the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ respectively, are applied on ΔABC shown in the diagram below



(a) On the same axes, draw the image of ΔABC under the transformation **P**.

(b) Explain clearly why applying these two transformations in a different order will result in different images.

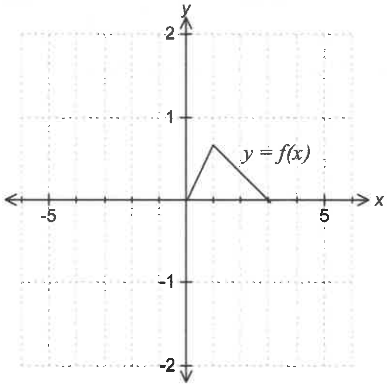
(c) Given that the area of ΔABC is k units^2, find the area of the image of ΔABC under the transformations **Q** followed by **P**.

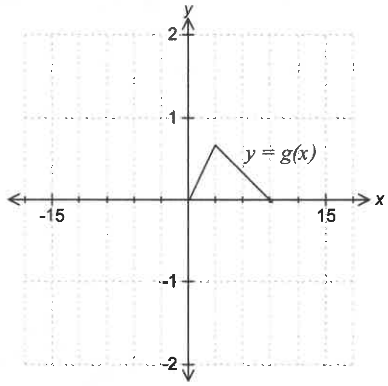
(d) State the matrix of one possible linear transformation **R**, such that the order of application **PR** and **RP** gives identical results.

Calculator Assumed

8. [5 marks: 2, 3]

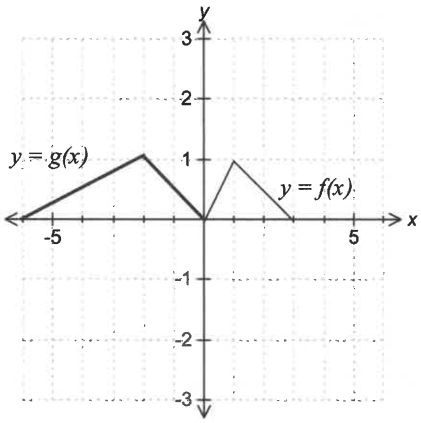
(a) The graphs of $y = f(x)$ and $y = g(x)$ are drawn below.





State the matrix representing the linear transformation that maps $y = f(x)$ to $y = g(x)$.

(b) The graphs of $y = f(x)$ and $y = g(x)$ are drawn below.



State the matrices representing the linear transformations that maps $y = f(x)$ to $y = g(x)$. Discuss the order in which these transformations are applied.

Calculator Assumed

9. [5 marks: 2, 3]

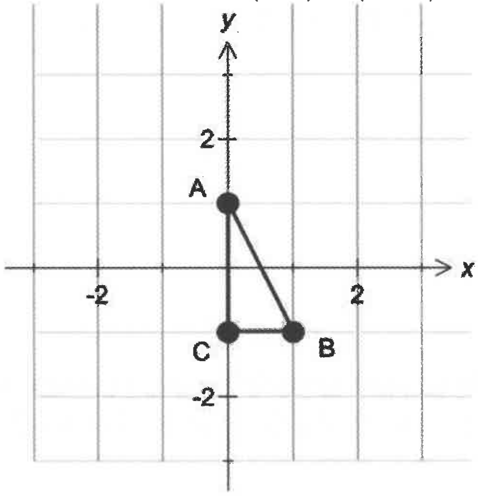
(a) Find the matrix representation of a combination of linear transformations that map the points (1, 0) to (–2, 0) and (2, 1) to (–3, 1).

(b) Show that it is not possible to find a combination of linear transformations that will map the points (1, 0) to (3, 0), (2, 1) to (4, 1) and (3, 1) to (5, 1).

Calculator Assumed

10. [10 marks: 4, 6]

The triangle ABC has vertices A(0, 1), B(1, –1) and C(0, –1).



(a) ΔABC is mapped to $\Delta A'B'C'$ by a transformation represented by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Draw and clearly label $\Delta A'B'C'$ on the diagram above.

(b) $\Delta A'B'C'$ is mapped to $\Delta A''B''C''$ by a transformation represented by the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The points A'', B'' and C'' have coordinates (2, 3), (–1, k) and (k, –3) respectively. Find matrix **M** and the value(s) of k.

29 Complex Numbers

Calculator Free

1. [7 marks: 1, 2, 2, 2]

Solve for x.

(a) $x^2 + 4 = 0$

(b) $x^2 + x + 4 = 0$

(c) $(x + 2)^2 + 9 = 0$

(d) $x^3 + 9x = 0$

---

2. [5 marks: 1, 4]

Factorise using both real and complex factors where appropriate:

(a) $x^2 + 16$

(b) $x^2 + 2x + 5$

Calculator Free

3. [5 marks: 2, 3]

Express in the form $a + bi$:

(a) $(1 + i\sqrt{2})^2$

(b) $\dfrac{2+3i}{1-i}$

---

4. [5 marks]

Given that $(a + b\,i)^2 = -15 + 8\,i$, find a and b where a and b are real non-zero integers.

Calculator Free

5. [7 marks: 1, 3, 3]

Let $u = 1 + 3\,i$, $v = -1 + i$ and $w = -2\,i$.

(a) Find $u + v$.

(b) Find $\dfrac{v}{w}$.

(c) Find $v \times \overline{u}$.

---

6. [8 marks: 4, 4]

Find $w = a + b\,i$ where a and b are real non-zero integers if:

(a) $\overline{w} = \dfrac{5}{w}$,

(b) $w^2 = -3 - 4\,i$

Calculator Free

7. [9 marks: 3, 3, 3]

Let the complex numbers $z_1 = 2 + k\,i$ and $z_2 = -5 + 12\,i$, where k is a *real number*.
Determine all possible values of k if:

(a) $[\,\boldsymbol{Im}(z_1)\,]^2 = \boldsymbol{Im}(z_2)$

(b) $z_1{}^2 = z_2$

(c) $\dfrac{6z_1}{z_2} = -i.$

---

8. [8 marks: 2, 6]

Let the complex numbers $u = 3 + k\,i$ and $v = k + 2\,i$, where k is a *real number*.
Determine all possible values of k if:

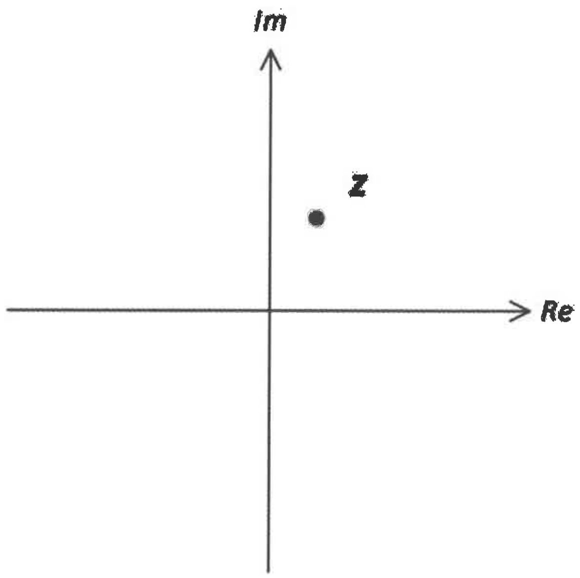
(a) $[\,\boldsymbol{Im}(u)\,]^2 = \boldsymbol{Re}(v)$

(b) $\dfrac{u}{v-1} = -1 - \dfrac{i}{2}$

Calculator Free

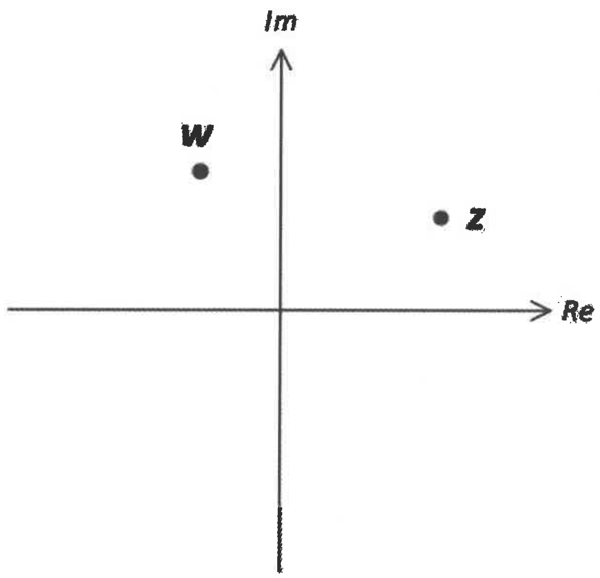
9. [3 marks]

The complex number z is plotted in the Argand plane shown below. In the Argand plane provided below, locate and label the points corresponding to: $\overline{z}$, $z+\overline{z}$ and $z-\overline{z}$.



10. [4 marks]

The complex numbers w and z are plotted in the Argand plane shown below. In the Argand plane provided below, locate and label the points corresponding to: $w+z$, $\overline{w+z}$, $w-z$ and $\overline{w-z}$.



30 Conjectures & Proofs

Calculator Free

1. [10 marks: 2, 2, 3, 3]

Provide a counter-example to disprove each of the following conjectures.

(a) If $x^2 > 100$, then $x > 10$.

(b) If $2 > x$, then $2^2 > x^2$.

(c) The sum of two numbers is always greater than the larger of the two numbers.

(d) The sum of any two positive numbers must always be less than the product of these two numbers.

Calculator Free

2. [5 marks: 1, 1, 3]

(a) Given that $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$. Find:

(i) $\mathbf{P} \times \mathbf{Q}$

(ii) $\mathbf{Q} \times \mathbf{P}$

(b) Prove or disprove the conjecture that if the matrices **P** and **Q** are such that $\mathbf{P} \times \mathbf{Q} = \mathbf{I}$, where I is the relevant identity matrix, then $\mathbf{P}^{-1} = \mathbf{Q}$.

Calculator Free

3. [9 marks: 3, 3, 3]

Provide a counter-example to disprove each of the following conjectures.

(a) Consider the parallelograms ABCD and PQRS. If the diagonals AC = PR then ABCD and PQRS are congruent.

(b) Consider the parallelograms ABCD and PQRS. If AB = PQ and the diagonals AC = PR, ABCD and PQRS are congruent.

(c) If the diagonals of a quadrilateral are perpendicular then the quadrilateral must be a rhombus.

Calculator Assumed

4. [12 marks: 2, 2, 4, 4]

Provide a counter-example to disprove each of the following conjectures.
Show all attempts, successful and otherwise.

(a) $2^{2n} + 1$ is prime for integer $n \geq 1$.

(b) $n^2 - n + 5$ is prime for integer $n \geq 1$.

(c) $1^n + 5^n + 10^n + 18^n + 23^n + 27^n = 2^n + 3^n + 13^n + 15^n + 25^n + 26^n$ for integer $n \geq 1$.

(d) There are no integer solutions to $a^3 + b^3 = c^3 + d^3$ where $a \neq b \neq c \neq d$.

Calculator Free

5. [3 marks]

Prove that $2^n - 1$ is always odd for integer $n \geq 1$.

---

6. [3 marks]

Prove that the square of an odd number add 11 is a multiple of 4.

---

7. [5 marks: 2, 3]

(a) Prove that product of three consecutive integers is divisible by 3.

(b) Prove that product of three consecutive integers is divisible by 6.

Calculator Assumed

8. [3 marks]

Prove that $x^7 - x$ is divisible by 6 for integer $x \geq 1$.

---

9. [4 marks]

Prove that the product of any three consecutive even numbers must be a multiple of 24.

---

10. [3 marks]

Prove that $4n^3 - 4n$ is a multiple of 24 for integer $n \geq 1$.

Calculator Assumed

11. [7 marks: 4, 3]

(a) It is conjectured that a number is divisible by 4 if the sum of twice the tens digit and the ones digit is a multiple of 4. Prove that this conjecture is true for a four digit number.

(b) Consider the positive integers a and b. The arithmetic mean of these two integers is $M = \dfrac{a+b}{2}$ and the harmonic mean is $H = \dfrac{2ab}{a+b}$.
It is conjectured that $M \geq H$. Use the expansion of $(\sqrt{a} - \sqrt{b})^2$ to prove this conjecture.

Calculator Assumed

12. [7 marks: 2, 5]

(a) Provide a counter-example to disprove the conjecture that the cube of an even number greater than two less the number itself is divisible by 5.

(b) Prove that $x^5 + x^4 + x^3 + x^2$ is divisible by 4 for x as a whole number.

Calculator Assumed

13. [7 marks: 1, 1, 5] [TISC]

Consider the conjecture that the product of any two positive numbers must be greater than the sum of these two numbers.

(a) Provide an example that proves this conjecture.

(b) Provide an example that disproves this conjecture.

(c) Under what conditions will this conjecture be always true?

Calculator Assumed

14. [7 marks: 2, 2, 3] [TISC]

Consider the conjecture that the *quotient* of any two positive numbers must be less than the *product* of these two numbers. That is: if one positive number is divided by another positive number, then the result must be less than the *product* of these two numbers.

(a) Provide an example that supports this conjecture.

(b) Provide an example that disproves this conjecture.

(c) Under what conditions will this conjecture be always true? Justify your answer.

Calculator Assumed

15. [10 marks: 3, 3, 4]

(a) Prove that $15.\overline{15}$ is a rational number.

(b) Prove that $5.7\overline{35}$ is a rational number.

(c) Prove that $10 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \ldots$ is a rational number.

Calculator Assumed

16. [9 marks: 5, 4]

(a) State the negative of the converse of the following conjecture:

If n is a multiple of 10, then n^2 is a multiple of 10.

Prove that the converse is true by proving that the negative of the converse is false.

(b) Use the result from (a) and the method of contradiction to prove that $\sqrt{10}$ is an irrational number.

Calculator Assumed

17. [10 marks: 5, 5]

(a) Consider the conjecture:

If m is a rational number and n is an irrational number,
then $m + n$ is an irrational number.

State the negative of this conjecture.
Prove that this conjecture is true by proving that its negative is false.

(b) Prove that if m is a rational number and n is an irrational number, then $m \times n$ is an irrational number.

Calculator Assumed

18. [7 marks: 5, 2]

(a) Use mathematical induction to prove that:

$$\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \ldots. + \frac{n}{5^n} = \frac{5}{16} - \frac{4n+5}{16(5^n)} \text{ for integer } n \geq 1.$$

(b) Discuss the sum as the number of terms increases indefinitely.

Calculator Assumed

19. [5 marks]

Prove inductively that $(1 + i)^{4n} = (-1)^n \, 2^{2n}$ for integer $n \geq 1$.

20. [5 marks]

Use mathematical induction to prove that $11^n + 4$ is divisible by 5 for integer $n \geq 1$.

Calculator Assumed

21. [6 marks]

Prove that $10^{n+1} + 3 \times 10^n + 5$ is a multiple of 9 for positive integer n.

22. [4 marks]

Given the non-singular commutative matrices **P** and **Q**, use mathematical induction to prove that for integer $n \geq 1$, $\mathbf{P}^n = \mathbf{Q}\,\mathbf{P}^n\,\mathbf{Q}^{-1}$.

Calculator Assumed

23. [5 marks] [TISC]

Use mathematical induction to prove that for integer $n \geq 1$:

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos (2n-1)x = \frac{\sin 2nx}{2 \sin x}.$$

[Hint: Use the formula $2 \cos \mathrm{A} \sin \mathrm{B} = \sin (\mathrm{A}+\mathrm{B}) - \sin (\mathrm{A}-\mathrm{B})$.]

Calculator Assumed

24. [4 marks]

Complete the table below.

Conjecture	
Negative of conjecture	
Contrapositive of conjecture	
Converse of conjecture	If x is an odd number then x^2 is an odd number.
Inverse of conjecture	

25. [5 marks: 2, 3]

Consider the conjecture:

If a is a factor of 20, then a is also factor of 40.

(a) Prove that the conjecture is true.

(b) State the inverse of the conjecture and determine with reasons whether it is true or false.

Calculator Assumed

26. [7 marks: 5, 2]

Consider the statement: $\tan\theta = 0 \quad \Rightarrow \quad \sin\theta = 0.$

(a) Determine with reasons if the contrapositive of this statement is true or false.

(b) Hence or otherwise, determine with reasons if the conjecture is true of false.

---

27. [6 marks: 3, 3]

Consider the statement: $\cos\theta = \dfrac{1}{2} \quad \Rightarrow \quad \tan\theta = \sqrt{3}$

(a) Determine with reasons if the converse of this statement is true or false.

(b) Hence, or otherwise determine with reasons if the inverse is true or false.

Fully Worked Solutions

01 Combinatorics I

Calculator Free

1. [9 marks: 2, 2, 2, 3]

Find k if:

(a) $\dfrac{10!}{k!} = 720$

$k! = \dfrac{10!}{720}$ ✓

$= \dfrac{10 \times 9 \times 8 \times 7!}{720} = 7!$

Hence, $k = 7$ ✓

(b) $\dfrac{k!}{8!} = 990$

$k! = 990 \times 8!$ ✓

$= 11 \times 10 \times 9 \times 8! = 11!$

Hence, $k = 11$ ✓

(c) ${}^{k}P_3 = 210$

$\dfrac{k!}{(k-3)!} = 210$ ✓

$k(k-1)(k-2) = 210$

Hence, $k = 7$ ✓

(d) $\dfrac{10! - k!}{k!} = 30\,239$

$\dfrac{k!\{[10 \times 9 \times 8 \times \ldots \times (k+1)] - 1\}}{k!} = 30\,239$ ✓

$10 \times 9 \times 8 \times \ldots \times (k+1) = 30\,240$ ✓

$9 \times 8 \times \ldots \times (k+1) = 3\,024$

$8 \times \ldots \times (k+1) = 336$

$7 \times (k+1) = 42$

$k = 5$ ✓

Calculator Free

2. [4 marks: 1, 1, 1, 1]

The letters of the word RICHES are rearranged in a line. No letter is used more than once. Write mathematical expressions for:

(a) the total number of possible arrangements.

> $6!$ ✓

(b) the number of arrangements with the letters I and E are adjacent.

> $5! \times 2$ ✓

(c) the number of arrangements with the letters I and E not adjacent

> $6! - 5! \times 2$ ✓

(d) the number of arrangements where the vowels are adjacent and the letters C and H are adjacent.

> $4! \times 2! \times 2!$ ✓

---

3. [4 marks: 1, 1, 1, 1]

Codes are formed by using the letters of the word MATHS, with no letter being used more than once:

(a) How many five letter codes start with M?

> Number $= 1 \times 4! = 24$ ✓

(b) How many five letter codes have the letters A and M together?

> Number $= 4! \times 2 = 48$ ✓

(c) How many four letter codes start with the letter M ?

> Number $= 1 \times 4 \times 3 \times 2 = 24$ ✓

(d) How many three letter codes end with the letter S.

> Number $= 4 \times 3 \times 1 = 12$ ✓

Calculator Assumed

4. [4 marks: 1, 1, 1, 1]

Three year 11 students and four year 12 students are to be arranged in a line.

(a) In how many ways can this be done?

> Number of ways $= 7! = 5040$ ✓

(b) In how many arrangements are all the year 11 students next to each other?

> $N = 5! \times 3! = 720$ ✓

(c) How many arrangements have all students of the same year group adjacent to each other?

> $N = 2! \times 3! \times 4! = 288$ ✓

(d) How many arrangements have no student of the same year group adjacent to each other?

> $N = 3! \times 4! = 144$ ✓

---

5. [9 marks: 1, 2, 3, 3]

Six digit numbers are formed using the digits 1, 2, 3, 4, 5, and 6. Each digit is used only once.

(a) How many six digit numbers are possible?

> $6! = 720$ ✓

(b) How many six digit even numbers are possible?

> $N = 5 \times 4 \times 3 \times 2 \times 1 \times 3 = 360$ ✓✓

(c) How many five digit even numbers greater than 40 000 are possible?

> If the first digit is odd: $N = 1 \times 4 \times 3 \times 2 \times 3 = 72$ ✓
> If the first digit is even: $N = 2 \times 4 \times 3 \times 2 \times 2 = 96$ ✓
> Hence, total $= 72 + 96 = 168$ ✓

(d) How many even numbers greater than 40 000 are possible?

> $N = n$(six digit even numbers > 40 000) + n(five digit even numbers > 40 000) ✓
> $= 360 + 168 = 528$ ✓✓

Calculator Assumed

6. [7 marks: 2, 2, 3]

Passwords consisting of between 8 and 12 characters inclusive are to be created using the letters of the alphabet (case sensitive) and the digits 0 to 9 inclusive.

(a) Write mathematical expressions for the number of possible passwords if:

(i) no character can be used more than once.

$N = {}^{62}P_8 + {}^{62}P_9 + {}^{62}P_{10} + {}^{62}P_{11} + {}^{62}P_{12}$ ✓✓

(ii) repetition of characters are permitted.

$N = 62^8 + 62^9 + 62^{10} + 62^{11} + 62^{12}$ ✓✓

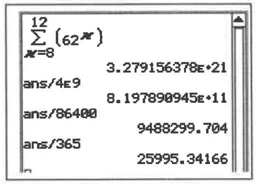
(b) A computer program is capable of checking 4 billion (1×10^9) passwords per second. How long will the computer take to check all the possible passwords in part (a) (ii)? Give your answer in years.

$N = 3.2792 \times 10^{21}$ ✓

$\text{Time} = \dfrac{3.2792 \times 10^{21}}{4 \times 10^9}$

$= 8.1979 \times 10^{11}$ seconds ✓

$\approx$ 25 995.34 years. ✓

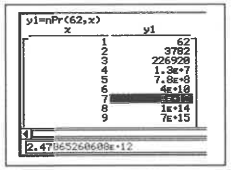


7. [4 marks: 2, 2]

The length of a password is determined by the number of characters in the password. Using the digits 0 to 9 inclusive and the case sensitive letters of the alphabet, find the minimum length of a password if the number of possible passwords is to exceed 1 trillion (1×10^{12}) if:

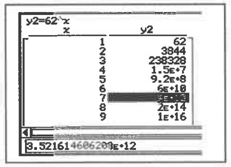
(a) repetition of characters is not permitted.

Use CAS table wizard:
${}^{62}P_6 = 4.4 \times 10^{10}$, ${}^{62}P_7 = 2.5 \times 10^{12}$ ✓
Hence, minimum length = 7 ✓



(b) repetition of characters is permitted.

Use CAS table wizard:
$62^6 = 5.7 \times 10^{10}$, $62^7 = 3.5 \times 10^{12}$ ✓
Hence, minimum length = 7 ✓



02 Combinatorics II

Calculator Assumed

1. [5 marks: 1, 2, 2]

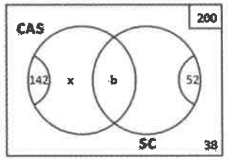
A random survey involving 200 students revealed the following. 38 students did not have any calculator (scientific or CAS) with them. 142 students had a CAS calculator with them. 52 students had a scientific calculator with them. How many students had:

(a) at least one calculator with them?

$n(\geq 1 \text{ calculator}) = 200 - 38 = 162$ ✓

(b) both types of calculators with them?

$n(CAS \cup SC) = n(CAS) + n(SC) - n(CAS \cap SC)$
$162 = 142 + 52 - n(CAS \cap SC)$ ✓
$n(CAS \cap SC) = 32$ ✓



(b) $38 + 52 + x = 200$
$x = 110$
$b = 142 - x$
$= 32$

(c) exactly one type of calculator with them?

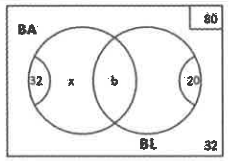
$n(\text{exactly one calculator}) = n(CAS \cup SC) - n(CAS \cap SC)$ ✓
$= 162 - 32$
$= 130$ ✓

2. [5 marks: 3, 2]

80 students were asked in a survey if they had previously broken a leg and/or broken an arm. Information from the survey indicated that 40% of students had suffered broken arms, one quarter of those surveyed had suffered broken legs and 40% had not previously had a broken arm or leg. How many students had:

(a) broken an arm and a leg?

$n(BA \cup \text{broken } BL) = 80 \times 0.6 = 48$ ✓
$n(BA \cup \text{broken } BL) = n(BA) + n(BL) - n(BA \cap BL)$
$48 = 32 + 20 - n(BA \cap BL)$ ✓
$n(BA \cap BL) = 4$ ✓



(a) $32 + 20 + x = 80$
$x = 28$
$b = 32 - x$
$= 4$

(b) broken an arm but not a leg?

$n(BA \cap \overline{BL}) = n(BA) - n(BA \cap BL)$ ✓
$= 32 - 4 = 28$ ✓

Calculator Assumed

3. [4 marks: 1, 1, 2]

In a group of 40 students, there were 10 boys who were colour vision deficient (CVD) and 15 girls who were not CVD. There were as many boys who were not CVD as there were boys who were CVD. How many of these students:

(a) were boys?

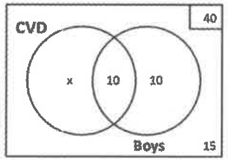
$n(\text{boys}) = n(\text{boys} \cap \text{CVD}) + n(\text{boys} \cap \overline{\text{CVD}})$
$= 10 + 10 = 20$ ✓

(b) were either boys or colour vision deficient?

$n(\text{boys} \cup \text{CVD}) = n(\text{total}) - n(\overline{\text{boys}} \cap \overline{\text{CVD}})$
$= 40 - 15 = 25$ ✓

(c) were colour vision deficient?

$n(\text{boys} \cup \text{CVD}) = n(\text{boys}) + n(\text{CVD}) - n(\text{boys} \cap \text{CVD})$
$25 = 20 + n(\text{CVD}) - 10$ ✓
$n(\text{CVD}) = 15$ ✓



(b) $15 + 10 + 10 + x = 40$
$x = 5$
$n(\text{boys} \cup \text{CVD})$
$= 10 + 10 + x = 25$

4. [5 marks: 2, 2, 1]

In a survey of teachers teaching mathematics to year nines, 15 teachers had a mathematics degree, 55 men teachers did not have a mathematics degree and 10 women teachers had a mathematics degree. There were as many teachers who were either female or had a mathematics degree as there were men teachers.

(a) How many male teachers were there?

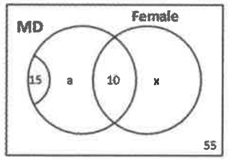
$n(\text{male}) = n(\text{male} \cap \text{MD}) + n(\text{male} \cap \text{no MD})$
$= (15 - 10) + 55 = 60$ ✓✓

(b) How many female teachers were there?

$n(\text{female} \cup \text{MD}) = n(\text{female}) + n(\text{MD}) - n(\text{female} \cap \text{MD})$
$60 = n(\text{female}) + 15 - 10$ ✓
$n(\text{female}) = 55$ ✓

(c) How many teachers were surveyed?

$n(\text{total}) = 55 + 60 = 115$ ✓



(a) $10 + a = 15$
$a = 5$
$n(\text{male}) = a + 55 = 60$
(b) $15 + x = 60$
$x = 45$
$n(\text{female}) = 10 + 45 = 55$

Calculator Assumed

5. [7 marks: 1, 1, 1, 1, 2, 1]

Flags of 10 different nations including that of Australia and New Zealand are to be flown from 10 flag poles set in a line. The flag poles are labelled poles 1 to 10. How many ways are there of assigning a flag to each of these poles if the:

(a) Australian flag must be flown from pole 1?

$N = 1 \times 9! = 362\,880$ ✓

(b) Australian flag or New Zealand flag must be flown from pole 1?

$N = 2 \times 9! = 725\,760$ ✓

(c) Australian flag and the New Zealand flag must be flown from poles 1 and 10 respectively?

$N = 1 \times 8! \times 1 = 40\,320$ ✓

(d) Australian flag must be flown from pole 1 and the New Zealand flag must not to be flown from pole 10.

$N = 1 \times 8! \times 8 = 322\,560$ ✓

(e) Australian flag must be flown from pole 1 or the New Zealand flag must be flown from pole 10.

$N = n(\text{Aussie flag on pole 1}) + n(\text{NZ flag on pole 10})$
$- n(\text{Aussie on pole 1} \cap \text{NZ on pole 10})$
$= 362\,880 + 362\,880 - 40\,320 = 685\,440$ ✓✓

(f) Australian flag must be flown from pole 1 or the New Zealand flag must be flown from pole 10 but not both at the same time.

$N = n(\text{Aussie on pole 1} \cup \text{NZ on pole 10}) - n(\text{Aussie on pole 1} \cap \text{NZ on pole 10})$
$= 685\,440 - 40\,320 = 645\,120$ ✓

Calculator Assumed

6. [6 marks: 2, 2, 2]

Twelve different coloured light bulbs, including a red bulb and a blue bulb are to be fitted into bulb sockets installed along a straight edge of a patio running East to West. The bulb socket at the extreme Eastern end is labelled E and the bulb socket at the extreme Western end is labelled W. Determine the number of arrangements with:

(a) the red light bulb not fitted into bulb sockets E or W.

N = 11 × 10! × 10 = 399 168 000 ✓✓

(b) the red light bulb and the blue light bulb not fitted into bulb sockets E or W.

N = 10 × 10! × 9 = 326 592 000 ✓✓

(c) the red light bulb or the blue light bulb not fitted into bulb sockets E or W.

N = n(red not in E or W) + n(blue not in E or W) − n(red not in E or W ∩ blue not in E or W)
= 399 168 000 + 399 168 000 − 326 592 000 ✓
= 471 744 000 ✓

7. [7 marks: 2, 2, 3]

Ten potted plants including four pots of roses of different shades of red and three pots of azaleas (each of a different colour) are to be arranged in a line along a footpath. How many arrangements will have:

(a) the potted roses adjacent to each other?

N = 7! × 4! = 120 960 ✓✓

(b) the potted azaleas adjacent to each other?

N = 8! × 3! = 241 920 ✓✓

(c) the roses adjacent to each other or the azaleas adjacent to each other?

N = n(roses adjacent) + n(azaleas adjacent) − n(roses adjacent ∩ azaleas adjacent)
= 120 960 + 241 290 − 5! × 4! × 3! ✓✓
= 345 600 ✓

Calculator Assumed

8. [6 marks: 2, 2, 2]

An analysis was conducted on the tickets bought by 150 patrons of the Perth International Festival. The survey was restricted to the three categories: Classical Music, Contemporary Music, Theatre, Circus & Dance (TCD).

- All had bought tickets to at least one of these categories.
- 30 had tickets to all categories.
- A total of 100 had tickets to a Classical Music event.
- A total of 90 had tickets to a Contemporary Music event.
- 80 had tickets to a Classical Music event as well as a TCD event.
- 50 had tickets to a Classical Music as well as a Contemporary Music event.
- 30 had tickets to a Contemporary Music as well as a TCD event.

(a) How many patrons had tickets to a Classical Music and a Contemporary Music event but not a Theatre, Circus & Dance event?

$n(\text{Classic} \cap \text{Contemporary} \cap \overline{\text{TCD}}) = n(\text{Classic} \cap \text{Contemporary}) - n(\text{Classic} \cap \text{Contemporary} \cap \text{TCD})$
$= 50 - 30 = 20$ ✓✓
Or Use a Venn Diagram (see below).

(b) How many patrons had tickets to only a Classical Music event?

$n(\text{Classic only}) = n(\text{Classic}) - n(\text{Classic} \cap \text{TCD}) - n(\text{Classic} \cap \text{Contemporary} \cap \overline{\text{TCD}})$
$= 100 - 80 - 20 = 0$ ✓✓
Or Use a Venn Diagram (see below).

(c) How many patrons had tickets to a Theatre, Circus & Dance event?

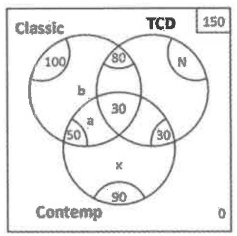
$n(\text{Classic} \cup \text{Contemporary} \cup \text{TCD}) = n(\text{Classic}) + n(\text{Contemporary}) + n(\text{TCD})$
$- n(\text{Classic} \cap \text{Contemporary}) - n(\text{Classic} \cap \text{TCD})$
$- n(\text{Contemporary} \cap \text{TCD})$
$+ n(\text{Classic} \cap \text{Contemporary} \cap \text{TCD})$
$150 = 100 + 90 + N - 50 - 80 - 30 + 30$ ✓
$N = 90$ ✓

Or Using a Venn Diagram.

(a) $a = 50 - 30 = 20$

(b) $b = 100 - 80 - a$
$= 100 - 80 - 20 = 0$

(c) $x = 90 - 30 - a$
$= 90 - 30 - 20 = 40$
Hence, $N = 150 - x - a - b$
$= 150 - 40 - 20 - 0 = 90$



Calculator Assumed

9. [6 marks: 3, 3]

A survey was conducted on 100 travellers at an airport.

- 62 had been vaccinated against yellow fever (Y).
- 80 had been vaccinated against typhoid (T).
- 82 had been vaccinated against malaria (M).
- 8 had not been vaccinated against any of these three diseases.
- All who had vaccinations had been vaccinated against more than one of these three diseases.

(a) How many travellers were vaccinated against yellow fever and typhoid but not malaria?

$n(Y \cup T \cup M) = 100 - 8 = 92$

$n(Y \cap \overline{T} \cap \overline{M}) = n(T \cap \overline{Y} \cap \overline{M}) = 0$ ✓

$n(Y \cup T \cup M) = n(M) + n(Y \cap T \cap \overline{M}) + n(Y \cap \overline{T} \cap \overline{M}) + n(T \cap \overline{Y} \cap \overline{M})$

$92 = 82 + n(Y \cap T \cap \overline{M}) + 0 + 0$ ✓

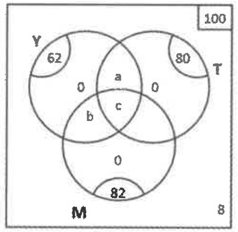
$n(Y \cap T \cap \overline{M}) = 10$ ✓

Or

From Venn Diagram:

$82 + 8 + a = 100$ ✓✓

$a = 10$ ✓



(b) How many travellers were vaccinated against yellow fever, typhoid and malaria?

$n(Y \cup T \cup M) = n(T) + n(Y \cap M \cap \overline{T}) + n(Y \cap \overline{T} \cap \overline{M}) + n(M \cap \overline{Y} \cap \overline{T})$

$92 = 80 + n(Y \cap M \cap \overline{T}) + 0 + 0$ ✓

$n(Y \cap M \cap \overline{T}) = 12$ ✓

$n(Y \cap T \cap M) = n(Y) - n(Y \cap T \cap \overline{M}) - n(Y \cap M \cap \overline{T})$

$= 62 - 10 - 12 = 40$ ✓

Or

From Venn Diagram:

$80 + 8 + b = 100$ ✓

$b = 12$ ✓

$a + b + c = 62$

$c = 62 - 10 - 12$

$= 40$ ✓

Calculator Assumed

10. [6 marks: 1, 4, 1]

50 students were asked if they had attended the open-day sessions at the University of Western Australia (W), Murdoch University (M) and Curtin University (C).

- 5 students did not attend any of these sessions.
- 20 students attended the open day session at W.
- 23 students attended the open day session at M.
- 30 students attended the open day session at C.
- 3 students attended the open day sessions at W and M but not at C.
- 7 students attended the open day sessions at C and M but not at W.
- 12 students attended the open day sessions at C and W but not at M.

(a) How many students attended sessions at exactly 2 of these universities?

N = 3 + 12 + 7 = 22 ✓

(b) How many students attended sessions at no more than 2 of these universities?

Let $n(W \cap M \cap C) = x$.

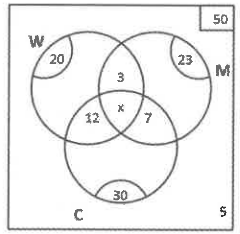
$n(W \cup M \cup C) = n(W) + n(M) + n(C) - n(W \cap M) - n(M \cap C) - n(C \cap W) + n(W \cap M \cap C)$

$50 - 5 = 20 + 23 + 30 - (3 + x) - (7 + x) - (12 + x) + x$ ✓✓

$x = 3$ ✓

Hence, $n(\leq 2) = n(W \cup M \cup C) - n(W \cap M \cap C)$

$= 45 - 3 = 42$ ✓



(c) How many students attended sessions at exactly one of these 3 universities?

$n(\text{exactly } 1) = n(\text{at least } 1) - n(\text{exactly } 2) - n(\text{exactly } 3)$

$= 45 - 22 - 3 = 20$ ✓

Calculator Assumed

11. [8 marks: 1, 1, 1, 2, 1, 2]

Seven students including Amy, Brian and Catherine are to be arranged in a line. How many possible arrangements are there with:

(a) Amy or Brian or Catherine on the extreme left?

$N = 3 \times 6! = 2\,160$ ✓

(b) Amy on the extreme left?

$n(\text{A left}) = 1 \times 6! = 720$ ✓

(c) Amy on the extreme left and Catherine on the extreme right?

$n(\text{A left} \cap \text{C right}) = 1 \times 1 \times 5! = 120$ ✓

(d) Amy on the extreme left or Catherine on the extreme right?

$$\begin{aligned} n(\text{A left} \cup \text{C right}) &= n(\text{A left}) + n(\text{C right}) - n(\text{A left} \cap \text{C right}) \\ &= 720 + 720 - 120 \quad ✓ \\ &= 1\,320 \quad ✓ \end{aligned}$$

(e) Amy on the extreme left, Brian in the middle and Catherine on the extreme right?

$n(\text{A left} \cap \text{B mid} \cap \text{C right}) = 1 \times 1 \times 1 \times 4! = 24$ ✓

(f) Amy on the extreme left or Brian in the middle or Catherine on the extreme right?

$$\begin{aligned} n(\text{A left} \cup \text{B mid} \cup \text{C right}) &= n(\text{A left}) + n(\text{B mid}) + n(\text{C right}) \\ &\quad - n(\text{A left} \cap \text{B mid}) - n(\text{A left} \cap \text{C right}) - n(\text{B mid} \cap \text{C right}) \\ &\quad + n(\text{A left} \cap \text{B mid} \cap \text{C right}) \\ &= 720 \times 3 - 120 \times 3 + 24 \quad ✓ \\ &= 1\,824 \quad ✓ \end{aligned}$$

Calculator Assumed

12. [11 marks: 1, 1, 2, 3, 2, 2]

Ten different coloured balls, including a red, a blue and a green ball, are to be placed in ten different boxes labelled A to J. One ball is to be placed in each box. How many ways are there of placing the balls (one in each box) with:

(a) the red ball in box A?

$n(\text{red in A}) = 1 \times 9! = 362\,880$ ✓

(b) the red ball in box A and the blue ball in box B?

$n(\text{red in A} \cap \text{blue in B}) = 1 \times 1 \times 8! = 40\,320$ ✓

(c) the red ball in box A or the blue ball in box B?

$$\begin{aligned} n(\text{red in A} \cup \text{blue in B}) &= n(\text{red in A}) + n(\text{blue in B}) - n(\text{red in A} \cap \text{blue in B}) \\ &= 362\,880 + 362\,880 - 40\,320 \quad ✓ \\ &= 685\,440 \quad ✓ \end{aligned}$$

(d) the red ball in box A or the blue ball in box B or the green ball in box C?

$$\begin{aligned} n(\text{red in A} \cup \text{blue in B} \cup \text{green in C}) &= n(\text{red in A}) + n(\text{blue in B}) + n(\text{green in C}) \\ &\quad - n(\text{red in A} \cap \text{blue in B}) - n(\text{red in A} \cap \text{green in C}) \\ &\quad - n(\text{blue in B} \cap \text{green in C}) \\ &\quad + n(\text{red in A} \cap \text{blue in B} \cap \text{green in C})) \\ &= 3 \times 362\,880 - 3 \times 40\,320 + 1 \times 1 \times 1 \times 7! \quad ✓✓ \\ &= 972\,720 \quad ✓ \end{aligned}$$

(e) the red ball in box A but the blue ball not in box B?

$n(\text{red in A} \cap \overline{\text{blue in B}}) = 1 \times 8 \times 8! = 322\,560$ ✓✓

(f) the green ball in box C but the red ball not in box A and the blue ball not in box B?

$$\begin{aligned} n(\text{green in C} \cap \overline{\text{red in A}} \cap \overline{\text{blue in B}}) &= n(\text{red in A} \cup \text{blue in B} \cup \text{green in C}) \\ &\quad - n(\text{red in A} \cup \text{blue in B}) \\ &= 972\,720 - 685\,440 \quad ✓ \\ &= 287\,280 \quad ✓ \end{aligned}$$

Calculator Assumed

13. [8 marks: 1, 1, 1, 1, 2, 2]

Consider the set of integers between 1 000 and 9 999 inclusive.
How many integers in this set:

(a) are divisible by 2?

> 1 000 = 2 × 500
> 9 998 = 2 × 4999
> Hence, n(divisible by 2) = 4 999 – 500 + 1 = 4 500 ✓

(b) are divisible by 3?

> 1 002 = 3 × 334
> 9 999 = 3 × 3 333
> Hence, n(divisible by 3) = 3 333 – 334 + 1 = 3 000 ✓

(c) are divisible by 2 and 3?

> 1 002 = 6 × 167
> 9 996 = 6 × 1 666
> Hence, n(divisible by 6) = 1 666 – 167 + 1 = 1 500 ✓

(d) are divisible by 2 and 6?

> n(divisible by 2 and 6) = n(divisible by 6)
> = 1 500 ✓

(e) are divisible by 2 or 3?

> n(divisible by 2 or 3) = n(divisible by 2) + n(divisible by 3) – n(divisible by 2 and 3)
> = 4 500 + 3 000 – 1 500 = 6 000 ✓✓

(f) are divisible by 2 or 3 but not both?

> n(divisible by 2 or 3 but not both) = n(divisible by 2 or 3) – n(divisible by 2 and 3)
> = 6 000 – 1 500 = 4 500 ✓✓

Calculator Assumed

14. [7 marks: 1, 1, 1, 2, 2]

Consider the set of integers between 500 and 5 000 inclusive.
How many integers in this set:

(a) are divisible by 5?

> 500 = 5 × 100
> 5 000 = 5 × 1 000
> Hence, n(divisible by 5) = 1 000 – 100 + 1 = 901 ✓

(b) are divisible by 10?

> 500 = 10 × 50
> 5 000 = 10 × 500
> Hence, n(divisible by 10) = 500 – 50 + 1 = 451 ✓

(c) are divisible by 5 and 10?

> LCM of 5 and 10 = 10
> Hence,
> n(divisible by 5 and 10) = n(divisible by 10)
> = 451 ✓

(d) are divisible by 5 or 10?

> n(divisible by 5 or 10) = n(divisible by 5) + n(divisible by 10) – n(divisible by 5 and 10)
> = 901+ 451 – 451 = 901 ✓✓
> OR
> All integers divisible by 10 are also divisible by 5.
> Hence, n(divisible by 5 or 10) = n(divisible by 5) = 901

(e) are divisible by 5 or 10 and is an even number?

> LCM of 2, 5 and 10 = 10
> Hence,
> n(divisible by 5 or 10 and is even) = n(divisible by 10) ✓
> = 451 ✓

Calculator Assumed

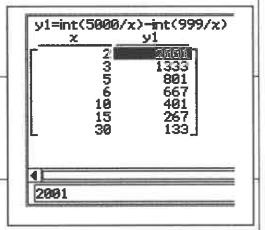
15. [6 marks: 3, 3]

Consider the set of integers between 1 000 and 5 000 inclusive.

(a) Complete the following table listing the number of multiples of n within this set for the given values of n.

– 1 per error ✓✓✓

n	Number of multiples of n in this set.
2	$\text{Int}\left(\frac{5000}{2}\right) - \text{Int}\left(\frac{999}{2}\right) = 2\,001$
3	$\text{Int}\left(\frac{5000}{3}\right) - \text{Int}\left(\frac{999}{3}\right) = 1\,333$
5	$\text{Int}\left(\frac{5000}{5}\right) - \text{Int}\left(\frac{999}{5}\right) = 801$
6	$\text{Int}\left(\frac{5000}{6}\right) - \text{Int}\left(\frac{999}{6}\right) = 667$
10	$\text{Int}\left(\frac{5000}{10}\right) - \text{Int}\left(\frac{999}{10}\right) = 401$
15	$\text{Int}\left(\frac{5000}{15}\right) - \text{Int}\left(\frac{999}{15}\right) = 267$
30	$\text{Int}\left(\frac{5000}{30}\right) - \text{Int}\left(\frac{999}{30}\right) = 133$



(b) Find the number of integers in this set that are multiples of 2, 3 or 5.

n(multiples of 2 or 3 or 5) = n(multiples of 2) + n(multiple of 3) + n(multiple of 5)
– n(multiples of 2 and 3) – n(multiples of 2 and 5)
– n(multiple of 3 and 5)
+ n(multiple of 2 and 3 and 5)
= 2 001 + 1 333 + 801 – 667 – 401 –267 + 133 ✓✓
= 2 933 ✓

Calculator Assumed

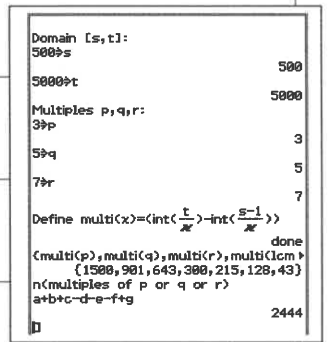
16. [6 marks: 3, 3]

Consider the set of integers between 500 and 5 000 inclusive.

(a) Complete the following table.

– 1 per error ✓✓✓

Multiples of	Number of multiples in this set.
3	$\text{Int}\left(\frac{5\,000}{3}\right) - \text{Int}\left(\frac{499}{3}\right) = 1\,500$
5	$\text{Int}\left(\frac{5\,000}{5}\right) - \text{Int}\left(\frac{499}{5}\right) = 901$
7	$\text{Int}\left(\frac{5\,000}{7}\right) - \text{Int}\left(\frac{499}{7}\right) = 643$
3 and 5	LCM of 3 and 5 = 15. $\Rightarrow \text{Int}\left(\frac{5\,000}{15}\right) - \text{Int}\left(\frac{499}{15}\right) = 300$
3 and 7	LCM of 3 and 7 = 21. $\Rightarrow \text{Int}\left(\frac{5\,000}{21}\right) - \text{Int}\left(\frac{499}{21}\right) = 215$
5 and 7	LCM of 5 and 7 = 35 $\Rightarrow \text{Int}\left(\frac{5\,000}{35}\right) - \text{Int}\left(\frac{499}{35}\right) = 128$
3 and 5 and 7	LCM of 3, 5 and 7 = 105. $\Rightarrow \text{Int}\left(\frac{5\,000}{105}\right) - \text{Int}\left(\frac{499}{105}\right) = 43$



(b) Find the number of integers in this set that are multiples of 3, 5 or 7.

n(multiples of 2 or 4 or 10) = n(multiples of 3) + n(multiple of 5) + n(multiple of 7)
– n(multiples of 3 and 5) – n(multiples of 3 and 7)
– n(multiple of 5 and 7)
+ n(multiple of 3 and 5 and 7)
= 1 500 + 901 + 643 – 300 – 215 –128 + 43 ✓✓
= 2 444. ✓

Calculator Assumed

17. [6 marks: 3, 3]

Consider the set of integers between 2500 and 10 000 inclusive.

(a) Complete the following table.

−1 per error ✓✓✓

Multiples of	Number of multiples in this set.
2	$\text{Int}\left(\frac{10\,000}{2}\right) - \text{Int}\left(\frac{2499}{2}\right) = 3\,751$
4	$\text{Int}\left(\frac{10\,000}{4}\right) - \text{Int}\left(\frac{2499}{4}\right) = 1\,876$
5	$\text{Int}\left(\frac{10\,000}{5}\right) - \text{Int}\left(\frac{2499}{5}\right) = 1\,501$
2 and 4	LCM of 2 and 4 = 4. Hence, 1 876.
2 and 5	LCM of 2 and 5 = 10. Hence, 751
4 and 5	LCM of 4 and 5 = 20 $\Rightarrow \text{Int}\left(\frac{10000}{20}\right) - \text{Int}\left(\frac{2499}{20}\right) = 376$
2 and 4 and 5	LCM of 2, 4 and 5 = 20. Hence, 376.

```
Domain [s,t]:
2500⇒s                                   2500
10000⇒t                                 10000
Multiples p,q,r:
2⇒p                                         2
4⇒q                                         4
5⇒r                                         5
Define multi(x)=(int(t/x)-int((s-1)/x))
                                         done
{multi(p),multi(q),multi(r),multi(lcm▸
 {3751,1876,1501,1876,751,376,376}
n(multiples of p or q or r)
a+b+c-d-e-f+g
                                         4501
```

(b) Find the number of integers in this set that are multiples of 2, 4 or 5.

n(multiples of 2 or 4 or 10) = n(multiples of 2) + n(multiple of 4) + n(multiple of 5)
− n(multiples of 2 and 4) − n(multiples of 2 and 5)
− n(multiple of 4 and 5)
+ n(multiple of 2 and 4 and 5)
= 3 751 + 1 876 + 1 501 − 1 876 − 751 −376 + 376 ✓✓
= 4 501 ✓

03 Combinatorics III

Calculator Free

1. [8 marks: 2, 2, 2, 2]

(a) Determine the $\binom{9}{6}$.

$\binom{9}{6} = \binom{9}{3} = \frac{9\times8\times7}{3\times2\times1} = 84$ ✓✓

(b) Evaluate $\binom{9}{3} + \binom{9}{4}$.

$\binom{9}{3} + \binom{9}{4} = 84 + \frac{9\times8\times7\times6}{4\times3\times2\times1}$
$= 84 + 126$
$= 210$ ✓✓

OR $\binom{9}{3} + \binom{9}{4} = \binom{10}{4}$ ✓
$= \frac{10\times9\times8\times7}{4\times3\times2\times1}$
$= 210$ ✓

(c) Simplify ${}^{10}C_5 \times {}^5P_5$. You are not required to evaluate your answer.

${}^{10}C_5 \times {}^5P_5 = \frac{10!}{5!\,5!} \times 5!$ ✓
$= \frac{10!}{5!}$ ✓
$= 10 \times 9 \times 8 \times 7 \times 6$

(d) Given that ${}^5P_{r+1} = {}^5P_r$, find r.

$\frac{5!}{(5-r-1)!} = \frac{5!}{(5-r)!}$
$\Rightarrow (4-r)! = (5-r)!$
$r = 4$ ✓✓

Calculator Free

2. [13 marks: 2, 2, 2, 4, 3]

(a) Find n and r if ${}^nC_r = \dfrac{n\times(n-1)\times 98}{3\times2\times1}$.

$n = 100$ ✓
$r = 3$ or 97 ✓

(b) Find n and r if ${}^nP_r = 20\times19\times18\times17$

$n = 20$ ✓
$r = 4$ ✓

(c) Find a and b if ${}^{30}C_a = {}^{3a}C_b$.

$3a = 30 \Rightarrow a = 10$ ✓
$b = 10$ or 20 ✓

(d) Find all possible values of a and b if ${}^{12}C_a = {}^{12}C_{2a+b}$.

$a + 2a + b = 12$ ✓

$a = 4 - \dfrac{b}{3}$

As a and b must be non-negative integers:
$b = 0, a = 4$ or $b = 3, a = 3$
or $b = 6, a = 2$ or $b = 9, a = 1$
or $b = 12, a = 0$ ✓✓✓

(e) Find a possible set of values for a and b if $10\times{}^9P_4 = 6\times{}^aP_b$.

$$10\times\frac{9!}{5!} = 6\times\frac{a!}{(a-b)!}$$
$$\frac{10\times9!}{6\times5!} = \frac{a!}{(a-b)!}$$ ✓
$$\frac{10!}{6!} = \frac{a!}{(a-b)!}$$
$\Rightarrow a = 10, b = 4$ ✓✓

Calculator Free

3. [3 marks]

(a) How many ways are there of choosing 2 students from a group of 100 students?

$N = {}^{100}C_2 = \dfrac{100\times99}{2\times1} = 4\,950$ ✓✓

(b) How many ways are there of choosing 98 students from a group of 100 students?

$N = {}^{100}C_{98} = {}^{100}C_2 = 4\,950$ ✓

---

4. [7 marks: 2, 2, 3]

A media folder has 10 video-clips. How many ways are there of choosing:

(a) seven of these clips?

$N = {}^{10}C_7 = {}^{10}C_3$
$= \dfrac{10\times9\times8}{3\times2\times1} = 120$ ✓✓

(b) seven or eight of these clips?

$N = {}^{10}C_7 + {}^{10}C_8$ ✓
$= {}^{11}C_8$
$= \dfrac{11\times10\times9}{3\times2\times1} = 165$ ✓

(c) at least one of these clips?

$N = {}^{10}C_1 + {}^{10}C_2 + \dots + {}^{10}C_{10}$ ✓
$= 2^{10} - 1$ ✓
$= 1023$ ✓

Calculator Assumed

5. [12 marks: 1, 3, 2, 3, 3]

A committee of 9 people are to be selected from 10 Labor, 8 Liberal and 5 Green politicians. How many different ways can the committee be selected if:

(a) there are no restrictions

No. of ways = $^{23}C_9$
= 817 190 ✓

(b) all three political parties are equally represented

No. of ways = $^{10}C_3 \times {}^{8}C_3 \times {}^{5}C_3$ ✓✓
= 67 200 ✓

(c) there are no Green representatives

No. of ways = $^{18}C_9$ ✓
= 48 620 ✓

(d) the Liberal representatives are in the (absolute) majority

No. of ways = $^{15}C_4 \times {}^{8}C_5 + {}^{15}C_3 \times {}^{8}C_6 + {}^{15}C_2 \times {}^{8}C_7 + {}^{15}C_1 \times {}^{8}C_8$ ✓✓
= 90 035 ✓

(e) a husband and wife pair, Alex and Alice, cannot be in the same committee.

No. of ways = $^{23}C_9 - {}^{2}C_2 \times {}^{21}C_7$ ✓✓
= 700 910 ✓

Calculator Assumed

6. [10 marks: 1, 2, 3, 4] [TISC]

Wei has a collection of 20 stickers in her pink box and 25 stickers in her blue box. All these stickers are different from each other.

(a) In how many ways can Wei pick 3 stickers from her pink box?

No. of ways = $^{20}C_3$
= 1140 ✓

(b) In how many ways can Wei pick 2 stickers from her blue box and arrange them in a line?

No. of ways = $^{25}C_2 \times 2!$ ✓
= 600 ✓

(c) In how many ways can Wei pick 3 stickers from her pink box and 2 stickers from her blue box and arrange them in a line if:

(i) there are no restrictions as to how the stickers are arranged?

No. of ways = $^{20}C_3 \times {}^{25}C_2 \times 5!$ ✓✓
= 41 040 000 ✓

(ii) all the stickers from the blue box must be together?

No. of ways = $^{20}C_3 \times {}^{25}C_2 \times 4! \times 2!$ ✓✓✓
= 16 416 000 ✓

Calculator Free

7. [12 marks: 1, 2, 2, 2, 2, 3]

In how many ways can the letters of the word **COMBINE** be arranged in a straight line (no letter may be used more than once):

(a) using all seven letters?

No. of ways = 7!
= 5040 ✓

(b) using all seven letters and starting with the letter **C**?

No. of ways = 1 × 6! ✓
= 720 ✓

(c) using all seven letters and ending in **BONE?**

No. of ways = 3! × 1 ✓
= 6 ✓

(d) using only 5 letters at a time?

No. of ways = ${}^7C_5 \times 5!$ ✓
= 2520 ✓

(e) using all seven letters with the vowels in the first three places (from the left)?

No. of ways = 3! × 4! ✓
= 144 ✓

(f) using only two vowels and two consonants?

No. of ways = ${}^3C_2 \times {}^4C_2 \times 4!$ ✓✓
= 432 ✓

Calculator Free

8. [12 marks: 1, 2, 2, 3, 4]

Consider the letters of the word CONQUEST. Write mathematical expressions for the number of ways the letters of this word can be *rearranged*:

(a) if the first letter must be a consonant.

N = 5 × 7! ✓

(b) if the first three letters must be consonants.

N = ${}^5C_3 \times 3! \times 5!$ ✓✓

(c) if one of the first three letters must be a consonant.

N = ${}^5C_1 \times {}^3C_2 \times 3! \times 5!$ ✓✓

(d) if at least one of the first three letters must be a consonant.

N = $({}^5C_1 \times {}^3C_2 \times 3! \times 5!) + ({}^5C_2 \times {}^3C_1 \times 3! \times 5!) + ({}^5C_3 \times {}^3C_0 \times 3! \times 5!)$ ✓ ✓ ✓

(e) if the vowels are to be sandwiched between two consonants (that is, a vowel must be preceded by a consonant and this vowel must also be followed by a consonant).

Let C: consonant and V: vowel.
The required arrangement must be of the form: CVCVCVCC or CCVCVCVC
or CVCCVCVC or CVCVCCVC ✓ ✓

Hence, N = 5! × 3! × 4 ✓✓

Calculator Free

9. [5 marks: 1, 1, 3] [TISC]

Jane wishes to create a password using the digits 1, 2, 3, 4, 5, 6 and the letters of her name J, A, N and E.

(a) How many eight character passwords can be created, if no character is used more than once?

> No. = ${}^{10}C_8 \times 8!$ ✓

(b) How many of the passwords in (a) include the letters J, A, N and E?

> No. = ${}^{6}C_4 \times 8!$ ✓

(c) How many passwords in (a) have digits adding up to exactly ten.

> N = N(digits 1, 2, 3, 4 and letters J, A, N, E) ✓
> = 8! ✓

---

10. [6 marks: 3, 3]

A number is divisible by four if the last two digits of the number is divisible by four. For example, 4564 is divisible by four because the last two digits "64" is divisible by four; but 4502 is not divisible by four because the last two digits "02" is not divisible by four. Using the digits 0 to 9 inclusive, write an expression for the number of five digit numbers divisible by four that can be formed:

(a) if digits may be repeated?

> Number of two digit numbers divisible by four
> = 25 ✓
> Hence, N = $9 \times 10 \times 10 \times 25$ ✓✓

(b) if no digit is to be repeated?

> Number of two digit numbers with no repeated digit divisible by four
> = $25 - n\{00, 44, 88\} = 25 - 3 = 22$ ✓
>
> Hence, N = $(8 \times 7 \times 6 \times 6)$ + $(7 \times 7 \times 6 \times 16)$ ✓✓
> (Zero in last 2 digits) (No Zero in last 2 digits)

Calculator Assumed

11. [8 marks: 1, 1, 2, 2, 2]

A contingent of six athletes is to formed from 10 swimmers, 12 cyclists and 8 gymnasts. How many different contingents can be formed if each contingent must have:

(a) 2 gymnasts?

> n(2 gymnasts) = ${}^{8}C_2 \times {}^{22}C_4$
> = 204 820 ✓

(b) 3 swimmers

> n(3 swimmers) = ${}^{10}C_3 \times {}^{20}C_3$
> = 136 800 ✓

(c) 2 gymnasts and 3 swimmers

> n(2 gymnasts ∩ 2 swimmers) = ${}^{8}C_2 \times {}^{10}C_3 \times {}^{12}C_1$
> = 40 320 ✓✓

(d) 2 gymnasts or 3 swimmers?

> n(2 gymnasts ∪ 3 swimmers) = n(2 gymnasts) + n(3 swimmers)
> − n(2 gymnasts ∩ 2 swimmers) ✓
> = 204 820 + 136 800 − 40 320
> = 301 300 ✓

(e) 2 gymnasts or 3 swimmers but not both?

> N = n(2 gymnasts ∪ 2 swimmers) − n(2 gymnasts ∩ 2 swimmers) ✓
> = 301 300 − 40 320
> = 260980 ✓

Calculator Assumed

12. [7 marks: 3, 2, 2]

Five books are to be selected and arranged on a library display shelf. These five books are to be selected from a collection of 10 adult novels, 5 non-fiction books and 8 illustrated children's books.

(a) Complete the table below.

Composition of books on display shelf	Number of different arrangements
2 adult novels	${}^{10}C_2 \times {}^{13}C_3 \times 5! = 1\,544\,400$ ✓
3 illustrated children's books	${}^{8}C_3 \times {}^{15}C_2 \times 5! = 705\,600$ ✓
2 adult novels and 3 illustrated children's book	${}^{10}C_2 \times {}^{8}C_3 \times 5! = 302\,400$ ✓

(b) Determine the number of possible arrangements with either 2 adult novels or 3 illustrated children's books.

$n(\text{2 novels} \cup \text{3 children's books}) = n(\text{2 novels}) + n(\text{3 children's books}) - n(\text{2 novels} \cap \text{3 children's books})$ ✓

$= 1\,544\,400 + 705\,600 - 302\,400$

$= 1\,947\,600$ ✓

(c) Determine the number of possible arrangements with either 2 adult novels or 3 illustrated children's books but not both.

$N = n(\text{2 novels} \cup \text{3 children's books}) - n(\text{2 novels} \cap \text{3 children's books})$ ✓

$= 1\,947\,600 - 302\,400$

$= 1\,645\,200$ ✓

Calculator Assumed

13. [8 marks: 3, 2, 3]

A password consisting of four characters is to be chosen from 10 digits, 26 upper case letters, 26 lower case letters and a set of 32 symbols.

(a) Complete the table below.

Composition of password	Number of different passwords
2 lower case letters	${}^{26}C_2 \times {}^{68}C_2 \times 4! = 17\,768\,400$ ✓
2 symbols	${}^{32}C_2 \times {}^{62}C_2 \times 4! = 22\,510\,464$ ✓
2 lower case letters and 2 symbols	${}^{26}C_2 \times {}^{32}C_2 \times 4! = 3\,868\,800$ ✓

(b) Determine the number of possible passwords with either two lower case letters or two symbols.

$n(\text{2 lower case} \cup \text{2 symbols}) = n(\text{2 lower case}) + n(\text{2 symbols}) - n(\text{2 lower case} \cap \text{2 symbols})$ ✓

$= 17\,768\,400 + 22\,510\,464 - 3\,868\,800$

$= 36\,410\,064$ ✓

(c) Determine the number of possible passwords with either two lower case letters or two upper case letters.

$n(\text{2 lower case} \cup \text{2 upper case}) = n(\text{2 lower case}) + n(\text{2 upper case}) - n(\text{2 lower case} \cap \text{2 upper case})$ ✓

$= 17\,768\,400 + 17\,768\,400 - {}^{26}C_2 \times {}^{26}C_2 \times 4!$ ✓

$= 17\,768\,400 + 17\,768\,400 - 2\,535\,000$

$= 33\,001\,800$ ✓

Calculator Assumed

14. [9 marks: 3, 2, 4]

A password consisting of four characters is to be chosen from 26 upper case letters, 26 lower case letters, 10 digits and a set of 32 symbols.

(a) Complete the table below.

Composition of password	Number of different passwords	
Exactly 2 lower case letters	${}^{26}C_2 \times {}^{68}C_2 \times 4! = 17\,768\,400$	✓
Exactly 1 symbol	${}^{32}C_1 \times {}^{62}C_3 \times 4! = 29\,045\,760$	✓
Exactly 2 lower case letters and 1 symbol	${}^{26}C_2 \times {}^{32}C_1 \times {}^{36}C_1 \times 4! = 8\,985\,600$	✓

(b) Determine the number of possible passwords with either two lower case letters or one symbol.

$n(\text{2 lower case} \cup \text{1 symbol}) = n(\text{2 lower case}) + n(\text{1 symbol})$
$\quad - n(\text{2 lower case} \cap \text{1 symbol})$ ✓
$= 17\,768\,400 + 29\,045\,760 - 8\,985\,600$
$= 37\,828\,560$ ✓

(c) Determine the number of possible passwords with either one symbol or one upper case letter.

$n(\text{1 symbol} \cup \text{1 upper case}) = n(\text{1 symbol}) + n(\text{1 upper case})$
$\quad - n(\text{1 symbol} \cap \text{1 upper case})$
$= 29\,045\,760 + {}^{26}C_1 \times {}^{68}C_3 \times 4! -$
$\quad - {}^{26}C_1 \times {}^{32}C_1 \times {}^{36}C_2 \times 4!$ ✓✓✓
$= 29\,045\,760 + 31\,272\,384 - 12\,579\,840$
$= 47\,738\,304$ ✓

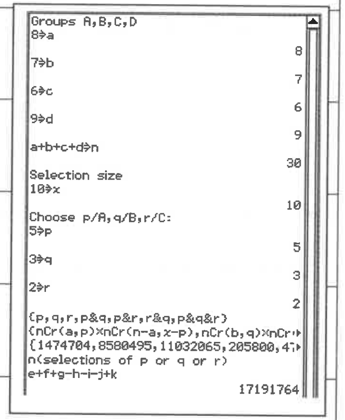
Calculator Assumed

15. [7 marks: 4, 3]

A sample of ten students is to be selected from a group of four year 8, five year 9, six year 10, seven year 11 and eight year 12 students.

(a) Complete the table below. ✓✓✓✓ –1 per error.

Composition of sample	Number of different samples
Five year 12 students.	${}^{8}C_5 \times {}^{22}C_5 = 1\,474\,704$
Three year 11 students.	${}^{7}C_3 \times {}^{23}C_7$ $= 8\,580\,495$
Two year 10 students.	${}^{6}C_2 \times {}^{24}C_8$ $= 11\,032\,065$
Five year 12 and three year 11 students.	${}^{8}C_5 \times {}^{7}C_3 \times {}^{15}C_2$ $= 205\,800$
Five year 12 and two year 10 students.	${}^{8}C_5 \times {}^{6}C_2 \times {}^{16}C_3$ $= 470\,400$
Three year 11 and two year 10 students.	${}^{7}C_3 \times {}^{6}C_2 \times {}^{17}C_5$ $= 3\,248\,700$
Five year 12 and three year 11 and two year 10 students.	${}^{8}C_5 \times {}^{7}C_3 \times {}^{6}C_2$ $= 29\,400$



(b) Determine the number of possible samples with five year 12 or three year 11 or two year 10 students.

$n(\text{5 yr 12} \cup \text{3 yr 11} \cup \text{2 yr 10}) = n(\text{5 yr 12}) + n(\text{3 yr 11}) + n(\text{2 yr 10})$
$\quad - n(\text{5 yr 12} \cap \text{3 yr 11}) - n(\text{5 yr 12} \cap \text{2 yr 10})$
$\quad - n(\text{3 yr 11} \cap \text{2 yr 10})$
$\quad + n(\text{5 yr 12} \cap \text{3 yr 11} \cap \text{2 yr 10})$ ✓
$= 1\,474\,704 + 8\,580\,495 + 11\,032\,065$
$\quad - 205\,800 - 470\,400 - 3\,248\,700$
$\quad + 29\,400$ ✓
$= 17\,191\,764$ ✓

04 Combinatorics IV

Calculator Free

1. [6 marks: 1, 2, 1, 2]

A container has 10 different sized pairs of nuts and bolts with the nuts removed from the respective bolts.

(a) Five bolts are randomly removed from this container. What is the minimum number of nuts that need to be removed from this container to ensure one matching pair of nut and bolt?

> Maximum number of incorrect nuts = 5
> Hence, minimum number of nuts = 5 + 1 = 6 ✓

(b) Six bolts are randomly removed from this container. What is the minimum number of nuts that need to be removed from this container to ensure two matching pairs of nut and bolt?

> Maximum number of incorrect nuts = 4
> Minimum number of nuts = 4 + 2 = 6 ✓✓

(c) What is the minimum number of items that need to be removed from this container to ensure one matching pair of nut and bolt?

> Minimum number of items = 10 + 1 = 11 ✓

(d) Fourteen items are randomly removed from the container.
(i) What is the minimum number of matching pairs of nuts and bolts?

> Minimum number of matching pairs = 14 – 10 = 4 ✓

(ii) What is the maximum possible number of matching pairs?

> Maximum possible number of matching pairs = $\frac{14}{2} = 7$ ✓

Calculator Free

2. [7 marks: 1, 2, 2, 2]

Twelve dog owners and their dogs (one dog per owner) meet at a dog park.

(a) Six dog owners are randomly chosen. What is the minimum number of dogs that need to be chosen to ensure a matching owner-dog pair?

> Maximum number of incorrect dogs = 6
> Hence, minimum number of dogs required = 6 + 1 = 7 ✓

(b) Seven dogs are randomly chosen. What is the minimum number of owners that need to be chosen to ensure three matching owner-dog pairs?

> Maximum number of incorrect owners = 5
> Hence, minimum number of owners required = 5 + 3 = 8 ✓✓

(c) A total of 15 owners and dogs were randomly selected.
(i) What is the minimum number of owner-dog pairs in this selection?

> Minimum number of owner-dog pairs = 15 – 12 = 3 ✓

(ii) What is the maximum possible number of owner-dog pairs in this selection?

> Maximum possible number of owner-dog pairs = $\frac{14}{2} = 7$. ✓

(d) If the owners came as couples, that is one dog per couple, what is the minimum number of persons and dogs that need to be chosen to ensure:
(i) a matching owner-dog pair?

> Minimum number of persons and dogs required = 24 + 1 = 25 ✓

(ii) more than two matching owner-dog pairs?

> Minimum number of persons and dogs required = 24 + 3 = 27 ✓

Calculator Free

3. [7 marks: 1 each]

A container has 5 red marbles, 6 green marbles and 9 yellow marbles. What is the minimum number of marbles that need to be drawn from this container to ensure:

(a) a marble of each colour?

Minimum number of marbles = 9 + 6 + 1 = 16 ✓

(b) two marbles of each colour?

Minimum number of marbles = 9 + 6 + 2 = 16 ✓

(c) two red marbles?

Minimum number of marbles = 9 + 6 + 2 = 17 ✓

(d) two yellow marbles?

Minimum number of marbles = 6 + 5 + 2 = 13 ✓

(e) two green marbles?

Minimum number of marbles = 9 + 5 + 2 = 16 ✓

(f) two marbles of the same colour?

Minimum number of marbles = 3 + 1 = 4 ✓

(g) three marbles of the same colour?

Minimum number of marbles = 6 + 1 = 7 ✓

Calculator Free

4. [7 marks: 1 each]

Dennis has 3 blue pens, 4 red pens and 5 black pens in his pencil case. What is the minimum number of pens that need to be drawn from the pencil case to ensure that:

(a) a red pen is drawn?

Minimum number of pens = 5 + 3 + 1 = 9 ✓

(b) a pen of each colour is drawn?

Minimum number of pens = 5 + 4 + 1 = 10 ✓

(c) a red and a blue pen is drawn?

Minimum number of pens = 5 + 4 + 1 = 10 ✓

(d) two blue pens and two red pens are drawn?

Minimum number of pens = 5 + 4 + 2 = 11 ✓

(e) two red pens and two black pens are drawn

Minimum number of pens = 3 + 5 + 2 = 10 ✓

(f) two blue pens and two black pens are drawn?

Minimum number of pens = 4 + 5 + 2 = 11 ✓

(g) two pens of the same colour are drawn?

Minimum number of marbles = 1 + 1 + 2 = 4 ✓

Calculator Assumed

5. [4 marks: 2, 2]

There are 25 students in a class.

(a) Explain clearly why there must be at least 3 students that are born in a same month.

$25 = 2 \times 12 + 1$.
In a worst case scenario, we could have the following distribution.

Birth month	Jan	Feb	Mar	…	Oct	Nov	Dec
No. of students	2 + 1	2	2	2	2	2	2

Hence, there are at least 3 students that are born in the same month. ✓✓

OR
Assume that there are no more than 2 students who share a same birth month. If this is the case, then, we have a total of $2 \times 12 = 24$ students which is one less than 25. Hence, there must be at least one month which is the birth month of at least 3 students. ✓✓

(b) Explain clearly why there must be at least one month which is a birth month shared by no more than 2 students.

$25 = 2 \times 12 + 1$.
In a worst case scenario, we could have the following distribution.

Birth month	Jan	Feb	Mar	…	Oct	Nov	Dec
No. of students	2 + 1	2	2	2	2	2	2

Hence, there is at least one month which is the birth month of no more than 2 students. ✓✓

OR
Assume that there is no month which is the birth month shared by no more than 2 students. That is, there are at least 3 students that have birthdays in each month. If this is the case, then, we have a total of $3 \times 12 = 36$ students which is far more than the 25 students in the class.. Hence, there must be at least one month which is the birth month of no more than 2 students. ✓✓

Calculator Assumed

6. [6 marks: 1, 1, 2, 2]

A class has 30 students.

(a) How many students need to be chosen to ensure that there are:
(i) two students who are born on the same day of the week?

Minimum number of students $= 7 + 1 = 8$ ✓

(ii) five students who are born on the same day of the week?

Minimum number of students $= 4 \times 7 + 1 = 29$ ✓

(b) There are at least x students who are born on the same day of the week. Find x. Justify your answer.

$30 = 4 \times 7 + 2$
In a worst case scenario, we could have the following distribution of birthdays:

Born on:	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
No. of students	5	5	4	4	4	4	4

Hence, there are at least 5 students who are born on the same day of the week. That is $x = 5$. ✓✓

(c) There must be at least one day of the week which is the birth day of no more than y students. Find y. Justify your answer.

$30 = 4 \times 7 + 2$
In a worst case scenario, we could have the following distribution of birthdays:

Born on:	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
No. of students	5	5	4	4	4	4	4

Hence, there is at least one day of the week which is the birth day of no more than 4 students. That is $y = 4$. ✓✓

Calculator Assumed

7. [7 marks: 1, 1, 1, 2, 2]

There are 300 students in a primary school.

(a) How many students need to be chosen to ensure that there are:
(i) two students with family names that start with the same letter?

Minimum number of students = 26 + 1 = 27 ✓

(ii) three students with family names that start with the same letter?

Minimum number of students = 26 × 2 + 1 = 53 ✓

(iii) six students with family names that start with the same letter?

Minimum number of students = 26 × 5 + 1 = 131 ✓

(b) There are at least x students with family names that start with the same letter. Find x. Justify your answer.

300 = 11 × 26 + 14.
In a worst case scenario, we could have the following distribution.
300 = 14 letters with 12 students each + 12 letters with 11 students each.
That is $x = 12$. ✓✓

(c) There must be at least one letter that is the first letter of the family names of no more than y students. Find y. Justify your answer.

300 = 11 × 26 + 14.
In a worst case scenario, we could have the following distribution:
300 = 14 letters with 12 students each + 12 letters with 11 students each.

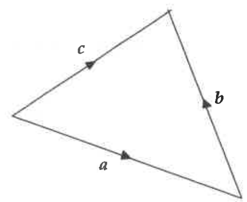
Hence, there is at least one letter that is the first letter of family names of no more than 11 students. That is $y = 11$. ✓✓

05 Addition & Subtraction of Vectors (Using Trigonometry)

Calculator Free

1. [2 marks: 1, 1]

Vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are as drawn in the accompanying diagram.



(a) Express $\boldsymbol{c}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ ✓

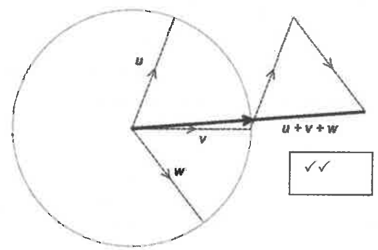
(b) Express $\boldsymbol{a}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$.

$\boldsymbol{a} = \boldsymbol{c} - \boldsymbol{b}$ ✓

2. [4 marks: 2, 2]

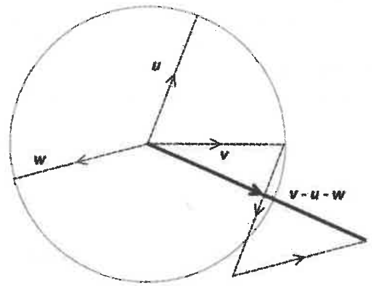
The accompanying diagram shows the relative positions of the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$. $|\boldsymbol{u}| = |\boldsymbol{v}| = |\boldsymbol{w}|$.

(a) Sketch in the space provided below $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$.



✓✓

(b) Sketch in the space provided below $\boldsymbol{v} - \boldsymbol{u} - \boldsymbol{w}$.



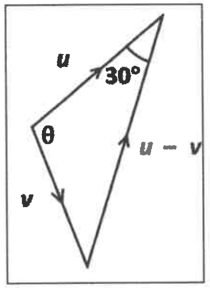
✓✓

Calculator Assumed

3. [7 marks: 3, 4]

The angle between u and $u - v$ is 30°, $|v| = 5$ and $|u - v| = 10$.

(a) Draw a clearly labelled sketch of the vectors u, v and $u - v$.



Angle between u and $u - v$ correct. ✓
Vectors correctly placed. ✓✓

(b) Use the rules of trigonometry to find $|u|$ and the angle between u and v.

Using the cosine rule:

$5^2 = 10^2 + u^2 - 2 \times 10 \times u \times \cos 30$ ✓

$u = 5\sqrt{3}$ ✓

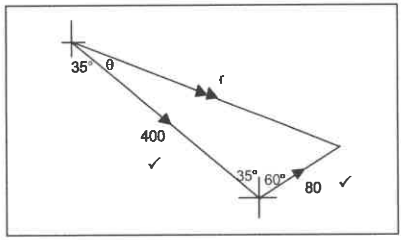
$\cos\theta = \dfrac{5^2 + (5\sqrt{3})^2 - 10^2}{2(5)(5\sqrt{3})}$ ✓

$\theta = 90^\circ$ ✓

4. [6 marks: 2, 4]

An aircraft is flying with a speed of 400 kmh^{-1} along bearing 145°. The aircraft is buffeted by a strong wind of magnitude 80 kmh^{-1} blowing from bearing 240°.

(a) Draw a sketch to indicate the actual direction of the aircraft.



Calculator Assumed

4. (b) Find the ground speed and direction of the aircraft.

$r^2 = 400^2 + 80^2 - 2(400(80)\cos 95$ ✓

$r = 414.7023$

Hence, ground speed of the aircraft is 414.7 kmh^{-1} ✓

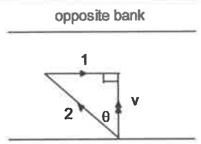
$\dfrac{\sin\theta}{80} = \dfrac{\sin 95}{414.7023} \Rightarrow \theta = 11.08^\circ$ ✓

Hence, true direction of aircraft is bearing 145° − 11.08° = 133.9°. ✓

5. [6 marks: 3, 3]

A boy intends to swim across a river of width 20 metres to the opposite bank. The river flows at a steady rate of 1 kmh^{-1}. The boy can swim at a steady speed of 2 kmh^{-1}.

(a) In what direction should the boy be headed so that he ends up at the opposite bank directly opposite to where he started off ?



$\sin\theta = ½ \Rightarrow \theta = 30^\circ$ ✓✓

Hence, the boy should be headed in a direction 60° with the near-bank upstream. ✓

(b) Find the time taken for the swim in part (a).

$v = \sqrt{(2^2 - 1^2)}$

$v = \sqrt{3}$ kmh^{-1} ✓

Hence, time taken $= \dfrac{0.020}{\sqrt{3}}$ ✓

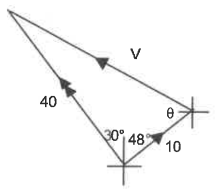
$= 0.01155$ hour

$= 41.6$ seconds ✓

Calculator Assumed

6. [5 marks]

A current is flowing in the direction N48°E at 10 kmh^{-1}. With what speed and in what direction should a naval vessel be travelling to achieve a resultant speed of 40 kmh^{-1} in the direction N30°W.



$v^2 = 40^2 + 10^2 - 2(40(10)) \cos 78$ ✓
$v = 39.1621$

Hence, the vessel should travel with a speed of 39.2 kmh^{-1} ✓

$\frac{\sin\theta}{40} = \frac{\sin 78}{39.1621} \Rightarrow \theta = 87.54^\circ$ ✓✓

Hence, the vessel should travel in the direction $(180 + 48 + 87.5)^\circ$, that is 315.5°. ✓

06 Components & Position Vectors I

Calculator Free

1. [8 marks: 2, 1, 1, 4]

Given that $\boldsymbol{a} = -2\,\mathbf{i} + 6\,\mathbf{j}$ and $\boldsymbol{b} = 5\,\mathbf{i} - 4\,\mathbf{j}$, find:

(a) $|\boldsymbol{a} + \boldsymbol{b}|$.

$|\boldsymbol{a}+\boldsymbol{b}| = |(-2\mathbf{i} + 6\mathbf{j}) + (5\mathbf{i} - 4\mathbf{j})|$ ✓
$= |3\mathbf{i} + 2\mathbf{j}|$
$= \sqrt{(3^2 + 2^2)}$
$= \sqrt{13}$ ✓

(b) the unit vector parallel to $\boldsymbol{a} + \boldsymbol{b}$.

Required unit vector $= \frac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$ ✓

(c) a vector that is parallel to $\boldsymbol{a} + \boldsymbol{b}$ but with a magnitude of 5.

Required vector $= 5 \times \frac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$
$= \frac{5\sqrt{13}}{13}(3\mathbf{i} + 2\mathbf{j})$ ✓

(d) $\boldsymbol{a}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ where $\boldsymbol{p} = 2\mathbf{i} + \mathbf{j}$ and $\boldsymbol{q} = -3\mathbf{i} + 2\mathbf{j}$.

Let $\boldsymbol{a} = \mu\boldsymbol{p} + \lambda\boldsymbol{q}$
$(-2\mathbf{i} + 6\mathbf{j}) = \mu(2\mathbf{i} + \mathbf{j}) + \lambda(-3\mathbf{i} + 2\mathbf{j})$
$-2\mathbf{i} + 6\mathbf{j} = (2\mu - 3\lambda)\,\mathbf{i} + (\mu + 2\lambda)\,\mathbf{j}$

Compare coefficients for i and j vectors:
$-2 = 2\mu - 3\lambda$ ✓
$6 = \mu + 2\lambda$ ✓

Solve simultaneously: $\mu = 2, \lambda = 2$

Hence, $\boldsymbol{a} = 2\boldsymbol{p} + 2\boldsymbol{q}$ ✓✓

Calculator Free

2. [6 marks]

OA = 3**i** + 10**j** , **OB** = 5**i** + b**j** and **OC** = 9**i** + c**j** .
Find c in terms of b if A, B and C are collinear.

AB = (5i + bj) – (3i + 10j)
= 2i + (b – 10) j ✓

AC = (9i + cj) – (3i + 10j)
= 6i + (c – 10) j ✓

For A, B and C to be collinear, **AB** = k**AC**.
Hence, 2i + (b – 10) j = k[6i + (c – 10) j] ✓

Compare i and j coefficients:
$2 = 6k \Rightarrow k = \frac{1}{3}$ ✓
$b - 10 = k(c - 10)$ ✓

Hence, $b - 10 = \frac{1}{3}(c - 10)$
$c = 3b - 20$ ✓

3. [7 marks]

Vector a**i** + (a + b)**j** has a magnitude of 5 and is parallel to vector 4**i** + 8**j** .
Find all possible values of a and b.

ai + (a + b)j = λ(4i + 8j) ✓

Compare coefficients for **i** and **j** vectors:
$a = 4\lambda \Rightarrow \lambda = \frac{a}{4}$ ✓
$a + b = 8\lambda$ ✓

Hence, $a + b = 2a \Rightarrow a = b$ ✓

Magnitude of ai + (a + b)j is 5.
Hence, $\sqrt{[a^2 + (a+b)^2]} = 5$ ✓
$\sqrt{(a^2 + 4a^2)} = 5$
$5a^2 = 25$
$a = b = \sqrt{5}$ ✓
$a = b = -\sqrt{5}$ ✓

Calculator Free

4. [5 marks]

Vector a **i** + 10**j** is of the same magnitude as (b – 10) **i** + (a – 2b) **j** but acts in the opposite direction. Find the values of a and b.

a i + 10j = – [(b – 10) i + (a – 2b) j]

a i + 10j = –(b – 10) i – (a – 2b) j ✓

Compare i and j coefficients:
$a = -b + 10 \Rightarrow a + b = 10$ ✓
$10 = -a + 2b \Rightarrow -a + 2b = 10$ ✓

Solve simultaneously: $a = \frac{10}{3}$, $b = \frac{20}{3}$ ✓✓

5. [4 marks]

Vector $\begin{pmatrix} a \\ b \end{pmatrix}$ has magnitude 20 and is parallel to $\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$.

Find the values of a and b.

$\left|\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}\right| = 2$ ✓

Hence, $\begin{pmatrix} a \\ b \end{pmatrix} = \pm 10 \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$. ✓

$\Rightarrow a = b = \pm 10\sqrt{2}$ ✓✓

6. [3 marks]

The point K divides the line segment AB internally in the ratio 4 : 1. Use a vector method to find the position vector of K if **OB** = – **i** + 2 **j** and **AB** = 15 **i** – 5 **j** .

OK = **OB** + **BK**
= **OB** + $\frac{1}{5}$**BA** ✓
= < –1, 2 > + $\frac{1}{5}$< –15, 5 > ✓
= < –4, 3 > ✓

Calculator Assumed

7. [3 marks]

The points P and Q have position vectors $5\mathbf{i} - 2\mathbf{j}$ and $-4\mathbf{i} + 5\mathbf{j}$ respectively.
The point K is such that $\mathbf{PK} = -4\mathbf{QK}$. Find the position vector of K.

$\mathbf{PK} = -4\mathbf{QK} \quad \Rightarrow \mathbf{PK} = 4\mathbf{KQ}$ ✓

Hence: $\mathbf{OK} - (5\mathbf{i} - 2\mathbf{j}) = 4[(-4\mathbf{i} + 5\mathbf{j}) - \mathbf{OK}]$ ✓

$5\mathbf{OK} = -11\mathbf{i} + 18\mathbf{j}$

$\mathbf{OK} = \frac{1}{5}(-11\mathbf{i} + 18\mathbf{j})$ ✓

OR

$\mathbf{OK} = \frac{1}{5}[4\mathbf{OQ} + \mathbf{OP}]$ ✓

$= \frac{1}{5}[4(-4\mathbf{i} + 5\mathbf{j}) + (5\mathbf{i} - 2\mathbf{j})]$ ✓

$= \frac{1}{5}(-11\mathbf{i} + 18\mathbf{j})$ ✓

8. [4 marks]

It is known that $\mathbf{OA} = a\mathbf{i} + \mathbf{j}$ and $\mathbf{OB} = 4\mathbf{i} + b\mathbf{j}$.
K is a point such that AK: AB = 2 : 5 and $\mathbf{OK} = 4\mathbf{i} - 3\mathbf{j}$. Find a and b.

AK:AB = 2:5 $\Rightarrow$ $\mathbf{AK} = \frac{2}{5}\mathbf{AB}$ ✓

$\mathbf{AB} = \begin{pmatrix}4\\b\end{pmatrix} - \begin{pmatrix}a\\1\end{pmatrix} = \begin{pmatrix}4-a\\b-1\end{pmatrix}$

$\mathbf{AK} = \begin{pmatrix}4\\-3\end{pmatrix} - \begin{pmatrix}a\\1\end{pmatrix} = \begin{pmatrix}4-a\\-4\end{pmatrix}$

Hence $\begin{pmatrix}4-a\\-4\end{pmatrix} = \frac{2}{5}\begin{pmatrix}4-a\\b-1\end{pmatrix}$ ✓

Compare i and j coefficients:

$4 - a = \frac{2}{5}(4-a) \quad \Rightarrow 4 - a = 0 \Rightarrow a = 4$ ✓

$-4 = \frac{2}{5}(b-1) \quad \Rightarrow b = -9$ ✓

Calculator Assumed

9. [5 marks]

Vector $\boldsymbol{u}$ has magnitude 100 kmh^{-1} and acts in the direction 040°.
Vector $\boldsymbol{v}$ has magnitude 150 kmh^{-1} and acts in the direction 280°.
Let $\boldsymbol{i}$ be the unit vector in the West-East direction
and $\boldsymbol{j}$ be the unit vector in the South-North direction.

Use vector components to find the magnitude and direction of $\boldsymbol{u} - 2\boldsymbol{v}$.

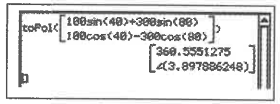
$\boldsymbol{u} = \begin{pmatrix}100\sin 40\\100\cos 40\end{pmatrix} \quad \boldsymbol{v} = \begin{pmatrix}-150\sin 80\\150\cos 80\end{pmatrix}$ ✓✓

$\boldsymbol{u} - 2\boldsymbol{v} = \begin{pmatrix}100\sin 40 + 300\sin 80\\100\cos 40 - 300\cos 80\end{pmatrix}$ ✓

$|\boldsymbol{u} - 2\boldsymbol{v}| = 360.6$ kmh^{-1} ✓

Direction 3.9° with the $\boldsymbol{i}$ vector. ✓

That is along bearing 086.1°.



10. [5 marks]

Given that $\boldsymbol{u} = <-4, 16>$ and $|\boldsymbol{v}| = 100$, find $\boldsymbol{v}$
if $\boldsymbol{u} + \boldsymbol{v}$ is to be in the same direction as the vector $<2, 2>$.

$\begin{pmatrix}-4\\16\end{pmatrix} + \boldsymbol{v} = \lambda\begin{pmatrix}2\\2\end{pmatrix}$ ✓

$\boldsymbol{v} = \lambda\begin{pmatrix}2\\2\end{pmatrix} - \begin{pmatrix}-4\\16\end{pmatrix}$

$= \begin{pmatrix}2\lambda+4\\2\lambda-16\end{pmatrix}$ ✓

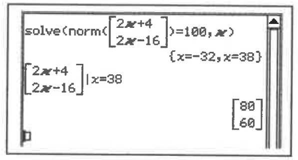
But $|\boldsymbol{v}| = 100$:

$\sqrt{(2\lambda+4)^2 + (2\lambda-16)^2} = 100$ ✓

$\lambda = -32, 38$ ✓

But $\lambda > 0$, hence, $\lambda = 38$

$\boldsymbol{v} = \begin{pmatrix}80\\60\end{pmatrix}$ ✓

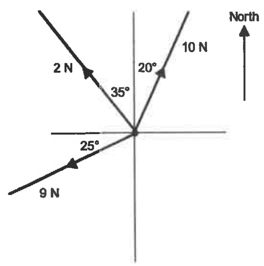


07 Components & Position Vectors II

Calculator Assumed

1. [6 marks: 3, 3]

The diagram below shows the forces acting on a body. The forces are all on the same plane.



(a) Find the magnitude of the resultant.

Vertical component of resultant
= 2 cos 35 + 10 cos 20 – 9 sin 25
= 7.2317 ✓

Horizontal component of resultant
= –2 sin 35 + 10 sin 20 – 9 cos 25
= –5.8837 ✓

Hence, magnitude of resultant
= $\sqrt{(7.2317^2 + 5.8837^2)}$
= 9.3228
= 9.32 N ✓

(b) Find the magnitude and the direction of a single force that will keep this system in equilibrium.

Resultant = – 5.8837 i + 7.2317 j

Hence, force required to maintain equilibrium
= –(– 5.8837 i + 7.2317 j)
= 5.8837 i – 7.2317 j ✓

Magnitude = 9.32 N ✓

$\tan\theta = \dfrac{7.2317}{5.8837}$

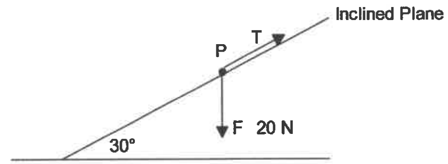
$\theta = 50.87°$

Hence, direction is 140.87°. ✓

Calculator Assumed

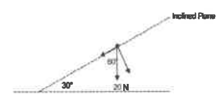
2. [5 marks: 2, 2, 1]

In the diagram below, a particle P is on a plane inclined at an angle of 30° to the horizontal. A vertical force F of magnitude 20 N is acting on P as shown. Force T parallel to the inclined plane is applied to prevent P from slipping down the inclined plane.



(a) Find the magnitude of the component of F parallel to the inclined plane.

Magnitude of component of F parallel to the plane
= 20 cos 60 = 10 N ✓✓



(b) Find the magnitude of the component of F perpendicular to the inclined plane.

Magnitude of component of F perpendicular to the plane
= 20 sin 60 = $10\sqrt{3}$ N ✓✓

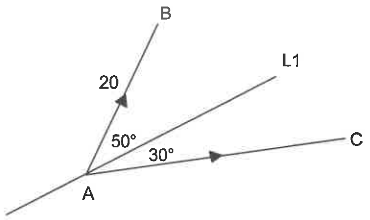
(c) Find the magnitude of force T.

Magnitude of T = magnitude of component of F parallel to the plane
= 10 N ✓

Calculator Assumed

3. [4 marks: 1, 1, 2]

In the diagram below, vector **AB** is of magnitude 20 units and is inclined at an angle of 50° to the line L1. Vector **AC** is inclined at angle of 30° to the line L1 as shown.



(a) Find the magnitude of the component of **AB** parallel to the line L1.

> Magnitude of component of AB parallel to L1
> = 20 cos 50
> = 12.8557
> = 12.86 units ✓

(b) Find the magnitude of the component of **AB** perpendicular to line L1.

> Magnitude of component of F perpendicular to L1
> = 20 sin 50
> = 15.3208
> = 15.32 units ✓

(c) Find the magnitude of **AC** if the resultant of the vectors **AB** and **AC** is parallel to the line L1.

> Magnitude of Component of **AC** perpendicular to L1
> = Magnitude of Component of **AB** perpendicular to L1
>
> Hence, AC sin 30 = 15.3208
> AC = 30.6416
> = 30.64 units ✓
>
> Hence, magnitude of **AC** is 30.64 units. ✓

Calculator Assumed

4. [11 marks: 1, 1, 4, 2, 3]

A light plane can fly at 80 km per hour in still air. The pilot wishes to fly from O to a neighbouring airstrip Q, located 40 km from O in the direction 060°. A constant wind of 20 km per hour is blowing from the North. **i** and **j** are unit vectors in the Easterly and Northerly directions respectively.

(a) Write in terms of **i** and **j** the position vector of Q relative to O.

> **OQ** = 40 sin 60 **i** + 40 cos 60 **j**
> $= 20\sqrt{3}\,\mathbf{i} + 20\,\mathbf{j}$ ✓

(b) Write in terms of **i** and **j** the velocity vector of the wind.

> $v = -20\,\mathbf{j}$ ✓

(c) Find the velocity vector the pilot should set so that the plane flies directly to Q.

> Let the velocity vector the pilot should set = v
> Resultant vector = $v + (-20\,\mathbf{j})$
>
> Hence: $v + (-20\,\mathbf{j}) = \lambda(20\sqrt{3}\,\mathbf{i} + 20\,\mathbf{j})$ where $\lambda > 0$ ✓
> $v = (20\lambda\sqrt{3}\,\mathbf{i} + (20\lambda + 20)\,\mathbf{j}$
>
> Since speed of plane = 80, $|v| = 80$
> Hence, $(20\sqrt{3}\,\lambda)^2 + (20\lambda + 20)^2 = 6400$ ✓
> $4\lambda^2 + 2\lambda - 15 = 0$
> $\lambda = 1.7026$ (reject -2.20 as $\lambda > 0$) ✓
>
> Hence, the velocity vector the pilot should set is 58.98 **i** + 54.05 **j** . ✓

(d) Find the resultant speed of the plane.

> Resultant velocity = $\lambda(20\sqrt{3}\,\mathbf{i} + 20\,\mathbf{j})$
> $= 1.7026(20\sqrt{3}\,\mathbf{i} + 20\,\mathbf{j})$ ✓
>
> Hence, resultant speed = $|1.7026(20\sqrt{3}\,\mathbf{i} + 20\,\mathbf{j})|$
> $= 68.104 \approx 68.1$ kmh^{-1} ✓

(e) Find the difference in flying time (to the nearest minute) caused by the wind.

> Travelling time in still air = $\frac{40}{80}$ = 0.5 hours = 30 minutes ✓
>
> Travelling time (with wind blowing) = $\frac{40}{68.104}$ = 0.5873 hours
> = 35.2 minutes ✓
> Hence, difference in travelling time = 35.2 – 30 = 9.5 = 5.2 minutes ✓

Calculator Assumed

5. [8 marks: 4, 2, 2]

A helicopter capable of flying at a speed of 100 km per hour in still air, takes off from O for a mining town located at A. The position vector of A relative to O is **200i** – **300j** km. Throughout the journey, the helicopter encounters a wind blowing with velocity **13i** + **5j** km per hour.

(a) Find the velocity vector the pilot should set so that the helicopter flies directly to A.

> Let the velocity vector be $\mathbf{v}$.
>
> Hence, for $\lambda > 0$:
> $(13\,\mathbf{i} + 5\,\mathbf{j}) + \mathbf{v} = \lambda\,(200\,\mathbf{i} - 300\,\mathbf{j})$ ✓
> $\mathbf{v} = (200\lambda - 13)\,\mathbf{i} - (300\lambda + 5)\,\mathbf{j}$
>
> Since speed of plane $|\mathbf{v}| = 100$,
> Hence, $(200\lambda - 13)^2 + (300\lambda + 5)^2 = 10\,000$ ✓
> $\lambda = 0.2832$ (reject -0.2663 as $\lambda > 0$) ✓
>
> Hence, the velocity vector the pilot should set is $43.645\,\mathbf{i} - 89.96\,\mathbf{j}$. ✓

(b) Find the time taken for the journey.

> Time taken $= \frac{1}{\lambda}$
> $= \frac{1}{0.2832}$ ✓
> $= 3.53107$ hours
> ≈ 3 hours 31.9 minutes ✓

(c) Find the velocity vector the pilot should set for a direct flight back to O. Assume that the wind blows with the same velocity throughout the flight back.

> For the return journey, $\lambda = -0.2663$ ✓
>
> Hence, the velocity vector the pilot should set is $-66.26\,\mathbf{i} + 74.89\,\mathbf{j}$. ✓

08 Components & Position Vectors III

Calculator Assumed

1. [6 marks: 1, 1, 1, 3]

A particle P, initially at **5i** – **10j** metres, moves with velocity **3i** + **4j** ms^{-1}.

(a) Find the position vector of P after 10 seconds.

> Position vector after 10 seconds
> $\mathbf{OP} = (5\mathbf{i} - 10\mathbf{j}) + 10(3\mathbf{i} + 4\mathbf{j})$
> $= 35\mathbf{i} + 30\mathbf{j}$ ✓

(b) Find the distance travelled by P after 10 seconds.

> Distance travelled $= |10(3\mathbf{i} + 4\mathbf{j})|$
> $= \sqrt{[(30)^2 + (40)^2]}$
> $= 50$ metres ✓

(c) Find the position vector of P after t seconds.

> $\mathbf{OP} = (5\mathbf{i} - 10\mathbf{j}) + t\,(3\mathbf{i} + 4\mathbf{j})$
> $= (5 + 3t)\,\mathbf{i} + (-10 + 4t)\,\mathbf{j}$ ✓

(d) When is P at a point with position vector (**65i** + **70j**) metres.

> $(5 + 3t)\,\mathbf{i} + (-10 + 4t)\,\mathbf{j} = 65\mathbf{i} + 70\mathbf{j}$ ✓
>
> Compare coefficients of **i** and **j** vectors:
> $5 + 3t = 65 \Rightarrow t = 20$
> $-10 + 4t = 70 \Rightarrow t = 20$ ✓
>
> Hence P is at point with position vector $(65\mathbf{i} + 70\mathbf{j})$ at $t = 20$ seconds. ✓

Calculator Assumed

2. [9 marks: 3, 3, 3]

The position vector of particles A and B, t hours after 12 noon, are
$\mathbf{r} = 12\mathbf{i} + 3\mathbf{j} + t\,(3\mathbf{i} + 4\mathbf{j})$ and $\mathbf{r} = -3\mathbf{i} - 5\mathbf{j} + t\,(2\mathbf{i} + 6\mathbf{j})$ metres respectively.

(a) Find in terms of t, the distance between A and B t hours after 12 noon.

> $\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$
> $= [(-3\mathbf{i} - 5\mathbf{j}) + t\,(2\mathbf{i} + 6\mathbf{j})] - [(12\mathbf{i} + 3\mathbf{j}) + t\,(3\mathbf{i} + 4\mathbf{j})]$
> $= (-15 - t)\,\mathbf{i} + (-8 + 2t)\,\mathbf{j}$ ✓✓
>
> Distance between A and B = $|\mathbf{AB}|$
> $= \sqrt{[(15 + t)^2 + (2t - 8)^2]}$
> $= \sqrt{[5t^2 - 2t + 289]}$ metres ✓

(b) Find when A and B are 18 metres apart.

> When $|\mathbf{AB}| = 18$, $5t^2 - 2t + 289 = 324$ ✓
> $t = 2.8533$ (reject -2.4533)
> = 2 hours 51 minutes ✓
> Hence, A and B are 18 metres apart at 2.51 pm. ✓

(c) Find when A is closest to B and find this distance.

> $|\mathbf{AB}| = \sqrt{[5t^2 - 2t + 289]}$ metres
> $|\mathbf{AB}|$ is minimized when $t = -\dfrac{(-2)}{2(5)}$ ✓
> = 0.2 hours after 12 noon ✓
> = 12 minutes after 12 noon
> Minimum distance = $\sqrt{[5(0.2)^2 - 2(0.2) + 289]} = 16.99$ m ✓
>
> OR
>
> fMin($\sqrt{5 \cdot t^2 - 2 \cdot t + 289}$, t)
> {MinValue=16.99411663, t=0.2} ✓
>
> $|\mathbf{AB}|$ is minimized at 12.12 pm with ✓
> minimum distance 16.99 m. ✓

Calculator Assumed

3. [9 marks: 2, 1, 2, 4]

The position vectors of A and B, t hours after 10 am are
$\mathbf{r} = -4\mathbf{i} - 4\mathbf{j} + t\,(2\mathbf{i} + 3\mathbf{j})$ and $\mathbf{r} = 3\mathbf{i} + 10\mathbf{j} + t\,(a\,\mathbf{i} + \mathbf{j})$ respectively.

(a) Find **AB** t hours after 10 am.

> $\mathbf{AB} = \mathbf{OB} - \mathbf{OA}$
> $= [(3\mathbf{i} + 10\mathbf{j}) + t\,(a\mathbf{i} + \mathbf{j})] - [(-4\mathbf{i} - 4\mathbf{j}) + t\,(2\mathbf{i} + 3\mathbf{j})]$
> $= [7 + (a-2)t]\,\mathbf{i} + (14 - 2t)\,\mathbf{j}$ ✓✓

(b) Find in terms of a and t, the distance between A and B, t hours after 10 am.

> Distance between A and B = $|\mathbf{AB}|$
> $= \sqrt{\{[7 + (a-2)t]^2 + [14 - 2t]^2\}}$ ✓

(c) Explain why when collision between A and B occurs, $\mathbf{AB} = 0\mathbf{i} + 0\mathbf{j}$

> When collision occurs, the A and B are in the same position.
> That is, **OA = OB**.
> Hence, $\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = 0\mathbf{i} + 0\mathbf{j}$. ✓✓

(d) Find the value of a if the two particles never collide.

> For the particles to collide $\mathbf{AB} = 0\mathbf{i} + 0\mathbf{j}$.
>
> Hence, $[7 + (a-2)t]\,\mathbf{i} + (14 - 2t)\,\mathbf{j} = 0\mathbf{i} + 0\mathbf{j}$ ✓
> $7 + (a - 2)\,t = 0$
> and $14 - 2t = 0$
> $\Rightarrow t = 7$ ✓
> Subst. $7 + (a - 2) \times 7 = 0 \Rightarrow a = 1$. ✓
>
> Hence, for A and B not to collide, $a \neq 1$. ✓

Calculator Assumed

4. [12 marks: 2, 4, 2, 4]

At 9.00 am, boat A is located at **2i** + 3**j** km and is travelling with velocity 2**i** – 3**j** km per hour. At 10.00 am, boat B is located at 2**i** + 3**j** km and is travelling with a speed of 5 km per hour. Let the velocity of boat B be x**i** + y**j** km per hour.

(a) Find the position vectors of A and B t hours after 10.00 am.

> Use 10.00 am as the reference time.
>
> **OA** = (2**i** + 3**j**) + (2**i** – 3**j**) + t (2**i** – 3**j**) = (4 + 2t) **i** – 3t **j** ✓
>
> **OB** = (2**i** + 3**j**) + t (x**i** + y**j**) = (2 + xt) **i** + (3 + yt) **j** ✓

(b) Show that when B intercepts A, $y = -1.5x$.

> **AB** = **OB** – **OA**
> = [(2 + xt) **i** + (3 + yt) **j**] – [(4 + 2t) **i** – 3t **j**]
> = [–2 + (x – 2) t] **i** + [3 + (3 + y) t] **j**
>
> At interception, **AB** = 0**i** + 0**j** .
> Compare coefficients of **i** and **j** vectors:
>
> $-2 + (x-2)t = 0 \Rightarrow t = \dfrac{2}{x-2}$ ✓
>
> $3 + (3+y)t = 0 \Rightarrow t = \dfrac{-3}{y+3}$ ✓
>
> Hence, $\dfrac{2}{x-2} = \dfrac{-3}{y+3} \Rightarrow 2y + 6 = -3x + 6$ ✓
>
> Therefore, $y = -1.5x$ ✓

(c) Hence, find the direction B should take in order to intercept A.

> Velocity vector of B = (x**i** + y**j**) .
>
> Direction B should take is $\tan^{-1}\left[\dfrac{y}{x}\right] = \tan^{-1}(-1.5) = 146.3^\circ$ ✓✓
>
> (Angle is in Quadrant 4 because $x > 2$ and $y < -3$.
> See expressions for t in part b.)

(d) Find when B intercepts A.

> Speed of boat = 5 $\Rightarrow x^2 + y^2 = 5^2$
> But $y = -1.5x$. $\Rightarrow x^2 + (1.5x)^2 = 25$ ✓
> $x = 2.7735$ (reject –2.7735 as $x > 2$) ✓
>
> Subst. $t = \dfrac{2}{2.7735 - 2} = 2.5856$ hours ✓
>
> Hence, B intercepts A at 12.35 pm. ✓

Calculator Assumed

5. [6 marks: 1, 1, 2, 2]

Particle P starts moving from the point A with position vector –2**i** + 3**j** metres with velocity **i** – 2**j** metres per second. Particle Q starts moving from the point A at the same time with velocity 2**i** + 3**j** metres per second.

(a) Determine the position vector of P after t seconds.

> **OA** = (–2**i** + 3**j**) + t (**i** – 2**j**)
> = (–2 + t) **i** + (3 – 2t) **j** ✓

(b) Determine the position vector of Q after t seconds.

> **OQ** = (–2**i** + 3**j**) + t (2**i** + 3**j**)
> = (–2 + 2t) **i** + (3 + 3t) **j** ✓

(c) Find in terms of t, the distance between P and Q after t seconds.

> **PQ** = **OQ** – **OP**
> = [(–2 + 2t) **i** + (3 + 3t) **j**] – [(–2 + t) **i** + (3 – 2t) **j**]
> = t **i** + 5t **j** ✓
>
> Distance |**PQ**| = $\sqrt{[t^2 + (5t)^2]}$
> = $t\sqrt{26}$ metres ✓

(d) Use your answer in (c) to find when P and Q are 10 metres apart.

> $t\sqrt{26} = 10$
> $t = 1.96$ seconds ✓✓

Calculator Assumed

6. [7 marks: 1, 2, 1, 3]

Particle P starts moving from the point with position vector 3**i** + 5**j** metres with velocity 2i – 3j metres per second.

(a) Determine the position vector of P after 3 seconds.

> **OP** = (3i + 5j) + 3 × (2i – 3j)
> = 9 **i** – 4 **j** ✓

(b) Find the distance from P to the point with position vector –2**i** + **j** after 3 seconds.

> Let fixed point be B.
>
> **PB** = **OB** – **OP**
> = [–2 **i** + **j**] – [9 **i** – 4 **j**]
> = –11 **i** + 5 **j** ✓
>
> Distance |**PB**| = $\sqrt{[11^2 + 5^2]}$
> = 12.08 metres ✓

(c) Determine the position vector of P after t seconds.

> **OP** = (3i + 5j) + t (2i – 3j)
> = (3 + 2t) **i** + (5 – 3t) **j** ✓

(d) Find when P is closest to the origin and state this distance.

> Distance |**OP**| = $\sqrt{[(3 + 2t)^2 + (5 - 3t)^2]}$
> = $\sqrt{(13t^2 - 18t + 34)}$ ✓
>
> Distance is minimum when $t = -\dfrac{-18}{2(13)} = 0.69$ ✓
>
> Hence, |**OP**| = 5.27 metres ✓
>
> OR
>
> fMin($\sqrt{(3+2t)^2+(5-3t)^2}$, t)
> {MinValue=5.269651864, t=0.6923076923}

Calculator Assumed

7. [9 marks: 2, 2, 2, 3]

A speed boat is moving at a constant velocity of 40 kmh^{-1} in the direction with bearing 060°. Initially, the speed boat is $5\sqrt{2}$ km from a buoy and is in the direction with bearing 225° from the buoy. Given that **i** and **j** are the unit vectors in the Easterly direction and Northerly direction respectively, find:

(a) the initial position vector of the speed boat with respect to the buoy in terms of **i** and **j**.

> **OP** = $-5\sqrt{2}$ sin 45 **i** – $5\sqrt{2}$ cos 45 **j**)
> = –5 **i** – 5 **j** ✓✓

(b) the direction vector of the speed boat in terms of **i** and **j**.

> v = 40 cos 30 **i** + 40 sin 30 **j**
> = $20\sqrt{3}$ **i** + 20 **j** ✓✓

(c) the position vector of the speed boat with respect to the buoy t hours later.

> $\boldsymbol{r}$ = (–5 **i** – 5 **j**) + t ($20\sqrt{3}$ **i** + 20 **j**)
> = (–5 + $20\sqrt{3}t$) **i** + (–5 + 20t) **j** ✓✓

(d) the time when the speed boat is nearest to the buoy and the least distance between the speed boat and the buoy.

> Distance to buoy = |$\boldsymbol{r}$|
> = $\sqrt{[(-5 + 20\sqrt{3}t)^2 + (-5 + 20t)^2]}$ ✓
> = $\sqrt{[1600t^2 - (200\sqrt{3} + 200)\,t + 50)]}$
>
> Distance is minimum when $t = -\dfrac{-(200\sqrt{3} + 200)}{2(1600)}$ = 0.1708 hours
> = 10.2 minutes ✓
>
> Hence, |$\boldsymbol{r}$| = 1.8301 km ✓
>
> Hence, speed boat is nearest the buoy after 10.2 minutes.
> Nearest distance = 1.83 km.
>
> OR
>
> fMin($\sqrt{(-5+20\sqrt{3}t)^2+(-5+20t)^2}$, t)
> {MinValue=1.830127019, t=0.1707531755}

Calculator Assumed

8. [9 marks: 4, 2, 3]

Yacht A starts sailing from the point L with position vector 5**i** – 2**j** metres with velocity –**i** + 3**j** . Yacht B starts sailing 10 seconds later with velocity 2**i** + **j** metres per second, from the point M with position vector 4**i** – 3**j** metres per second. t is time in seconds from the moment B starts sailing.

(a) Find in terms of t, the distance between A and B after t seconds.

OA = (5i – 2j) + (t +10) (–i + 3j)
= (–5 – t) i + (28 + 3t) j ✓

OB = (4i – 3j) + t (2i + j)
= (4 + 2t) i + (–3 + t) j ✓

AB = **OB** – **OA**
= (9 + 3t) i + (–31 – 2t) j ✓

Distance between A and B = |**AB**|
= $\sqrt{13t^2 + 178t + 1042}$ ✓

(b) When will A and B be 40 metres apart?

$\sqrt{13t^2 + 178t + 1042} = 40$ ✓
$t = 2.63$ (reject –16.3 as $t > 0$) ✓
A and B will be 40 metres apart 2.6 seconds after B starts moving.

(c) When will the two yachts be closest together? State this distance.

Use *f Min*($\sqrt{13t^2 + 178t + 1042}$, t, 0, ∞) ✓

Hence, yachts will be closest together before B starts moving. ✓
Closest distance = 32.5 metres. ✓

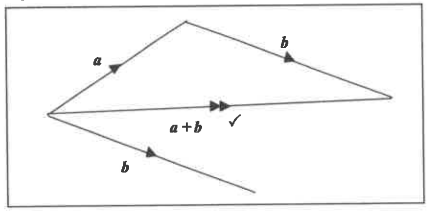
```
fMin(√(13t^2+17+1042) ,t,0,∞)
        {MinValue=32.54228019,t=0}
```


09 Relative Displacement

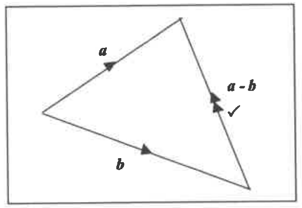
Calculator Free

1. [4 marks: 1, 1, 1, 1]

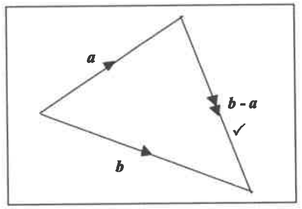
(a) Indicate clearly in the diagram below $\boldsymbol{a} + \boldsymbol{b}$.



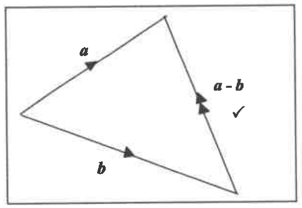
(b) Indicate clearly in the diagram below $\boldsymbol{a} - \boldsymbol{b}$.



(c) Indicate clearly in the diagram below the vector $\boldsymbol{b}$ relative to $\boldsymbol{a}$.



(d) Indicate clearly in the diagram below the vector $\boldsymbol{a}$ relative to $\boldsymbol{b}$.



Calculator Free

2. [4 marks: 1, 1, 2]

The position vectors of the points A, B, and C with respect to the origin O are **i** + **j**, 2**i** – **j** and –4**i** + 5**j** respectively.

(a) Find the position vector of A relative to B.

> ${}_{A}r_{B}$ = **BA** = **OA** – **OB**
> = (i + j) – (2i – j)
> = –i + 2j ✓

(b) Find the displacement of C relative to A.

> **AC** = **OC** – **OA**
> = (–4i + 5j) – (i + j)
> = –5i + 4j ✓

(c) The position vector of the point D relative to C is –2**i** – 3**j** .
Find the position vector of D relative to O.

> Position vector of D relative to C, **CD** = –2i – 3j
>
> But **CD** = **OD** – **OC** .
> ⇒ –2i – 3j = **OD** – (–4i + 5j) ✓
> ⇒ **OD** = (–2i – 3j) + (–4i + 5j)
> = –6i + 2j ✓

3. [2 marks: 1, 1]

The position vectors of the yachts WAone and WAtwo are 20**i** + 40**j** km and –10**i** + 50**j** km respectively.

(a) Find the position vector of WAtwo relative to WAone.

> Let P: WAone and Q: WAtwo
> **PQ** = **OQ** – **OP**
> = (–10i + 50j) – (20i + 40j)
> = –30i + 10j ✓

(b) Hence, find the distance between the two yachts.

> Distance = $\sqrt{(30^2 + 10^2)}$ ✓
> = $10\sqrt{10}$ km

Calculator Assumed

4. [5 marks: 2, 1, 2]

The position vector of Peter relative to a flag pole is 20**i** + 40**j** metres. Relative to Peter, Joe has position vector 5**i** – 15**j** metres.

(a) Find the position vector of Joe relative to the flagpole.

> Let O: the flagpole, J : Joe and P: Peter
>
> Position vector of Joe relative to Peter, **PJ** = 5i – 15j
> But **PJ** = **OJ** – **OP** .
> ⇒ (5i – 15j) = **OJ** – (20i + 40j) ✓
> ⇒ **OJ** = (5i – 15j) + (20i + 40j)
> = 25i + 25j ✓

(b) Hence, find the distance between Joe and the flag pole.

> Distance = $\sqrt{(25^2 + 25^2)}$
> = 35.4 m ✓

(c) The position vector of Kelly relative to Joe is a**i** + 20**j** metres.
Find the value of a if the distance between Kelly and Joe is 50 metres.

> **JK** = ai + 20j
>
> Hence, $a^2 + 20^2 = 50^2$ ✓
> $a = \pm 45.8$ m ✓

5. [4 marks]

Vectortown has position vector –20**i** + 10**j** km. The position vector of Trigtown relative to Vectortown is –40**i** – 15**j** km. The position vector of Easytown relative to Trigtown is 4**i** + 70**j** km. Find the position vector of Easytown.

> Let V: Vectortown and T: Trigtown
> **OV** = –20i + 10j
> Position vector of T relative to V, **VT** = –40i – 15j
> Position vector of E relative to T, **TE** = 4i + 70j
>
> **VT** = **OT** – **OV** ⇒ **OT** = **VT** + **OV**
> = (–40i – 15j) + (–20i + 10j)
> = –60i – 5j ✓✓
>
> Using **TE** = **OE** – **OT**
> ⇒ **OE** = **TE** + **OT**
> = (4i + 70j) + (–60i – 5j)
> = –56i + 65j ✓✓

10 Relative Velocity

Calculator Assumed

1. [7 marks: 1, 1, 3, 2]

Relative to an observer at O, A is moving with velocity $6\mathbf{i} + 9\mathbf{j}$ ms^{-1} and B is moving with velocity $-3\mathbf{i} + 4\mathbf{j}$ ms^{-1}.

(a) Find the velocity of A relative to B

$${}_A v_B = v_A - v_B$$
$$= (6\mathbf{i} + 9\mathbf{j}) - (-3\mathbf{i} + 4\mathbf{j})$$
$$= 9\mathbf{i} + 5\mathbf{j} \quad ✓$$

(b) What is the speed of A relative to B?

$$\text{Speed} = \sqrt{(9^2 + 5^2)}$$
$$= 10.3 \text{ ms}^{-1}. \quad ✓$$

(c) The velocity of C relative to B is $4\mathbf{i} - 5\mathbf{j}$ ms^{-1}. Find the velocity of C relative to A.

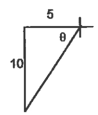
$${}_C v_B = v_C - v_B \Rightarrow v_C = {}_C v_B + v_B$$
$$= (4\mathbf{i} - 5\mathbf{j}) + (-3\mathbf{i} + 4\mathbf{j})$$
$$= \mathbf{i} - \mathbf{j} \quad ✓✓$$

$${}_C v_A = v_C - v_A$$
$$= (\mathbf{i} - \mathbf{j}) - (6\mathbf{i} + 9\mathbf{j})$$
$$= -5\mathbf{i} - 10\mathbf{j} \quad ✓$$

(d) In what direction is C moving relative to A?

${}_C v_A = -5\mathbf{i} - 10\mathbf{j}$

In the triangle sketched,
$\tan\theta = 2$
$\Rightarrow \theta = 63.4°$ ✓



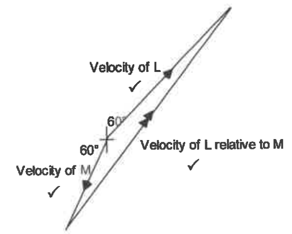
Hence, direction of C relative to A is along bearing $270° - 63.4° = 206.6°$. ✓

Calculator Assumed

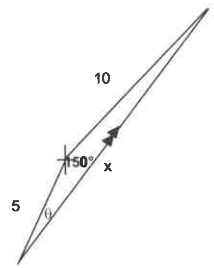
2. [7 marks: 3, 4]

L is travelling along bearing 060° with a speed of 10 kmh^{-1}. M is travelling along bearing 210° with a speed of 5 kmh^{-1}.

(a) Draw a clearly labelled vector diagram indicating the velocity vector of L relative to M.



(b) Use trigonometry to find the speed and direction of L relative to M.



In the diagram above:
$$x^2 = 10^2 + 5^2 - 2(10)(5)\cos 150$$
$$x = 14.5466 \quad ✓$$

Also: $\dfrac{\sin\theta}{10} = \dfrac{\sin 150}{14.5466}$
$\Rightarrow \sin\theta = 0.3437$
$\theta = 20.1°$ ✓✓

Hence, relative to M, L is travelling at a speed of 14.5 kmh^{-1} along the bearing 050.1°. ✓

Calculator Assumed

3. [7 marks: 2, 2, 1, 2]

P is travelling along bearing 045° with a speed of 20 kmh^{-1}. Q is travelling along bearing 240° with a speed of 15 kmh^{-1}. **i** and **j** are unit vectors in the Easterly and Northerly directions respectively.

(a) Express the velocities of P and Q in terms of **i** and **j**.

$v_P = 20 \cos 45\, \mathbf{i} + 20 \sin 45\, \mathbf{j}$
$= 10\sqrt{2}\, \mathbf{i} + 10\sqrt{2}\, \mathbf{j}$ ✓

$v_Q = -15 \cos 30\, \mathbf{i} - 15 \sin 30\, \mathbf{j}$
$= -\frac{15\sqrt{3}}{2}\, \mathbf{i} - \frac{15}{2}\, \mathbf{j}$ ✓

(b) Find the velocity of Q relative to P.

${}_Qv_P = v_Q - v_P$
$= (-\frac{15\sqrt{3}}{2}\, \mathbf{i} - \frac{15}{2}\, \mathbf{j}) - (10\sqrt{2}\, \mathbf{i} + 10\sqrt{2}\, \mathbf{j})$
$= -27.1325\mathbf{i} - 21.6421\mathbf{j}$
$= -27.1\mathbf{i} - 21.6\mathbf{j}$ ✓✓

(c) What is the speed of Q relative to P?

Speed of Q relative to P = $\sqrt{[(-27.1325)^2 + (-21.6421)^2]}$
$= 34.7$ kmh^{-1} ✓

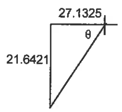
(d) What is the direction of Q relative to P?

${}_Qv_P = -27.1325\mathbf{i} - 21.6421\mathbf{j}$

In the triangle sketched,

$\tan\theta = \frac{21.6421}{27.1325}$

$\Rightarrow \theta = 38.6°$ ✓



Hence, direction of Q relative to P
is along bearing 270° – 38.6° = 231.4°. ✓

11 Relative Vectors

Calculator Assumed

1. [5 marks: 2, 3]

James is running along bearing 050° with a speed of 5 ms^{-1}. Wesley is running along bearing 300° with speed 4 ms^{-1}. **i** and **j** are unit vectors in the Easterly and Northerly directions respectively.

(a) Find in component form the velocity of Wesley relative to James.

Velocity of James $v_J = 5 \sin 50\, \mathbf{i} + 5 \cos 50\, \mathbf{j}$
Velocity of Wesley $v_W = -4 \sin 60\, \mathbf{i} + 4 \cos 60\, \mathbf{j}$
Hence, velocity of Wesley relative to James
${}_Wv_J = v_W - v_J$
$= (-4 \sin 60\, \mathbf{i} + 4 \cos 60\, \mathbf{j}) - (5 \sin 50\, \mathbf{i} + 5 \cos 50\, \mathbf{j})$
$= -7.2943\mathbf{i} - 1.2139\mathbf{j}$
$= -7.3\mathbf{i} - 1.2\mathbf{j}$ ✓✓

(b) Find how fast and in what direction is Wesley moving away from James.

${}_Wv_J = -7.3\mathbf{i} - 1.2\mathbf{j}$
= Polar[7.3946, –170.6°]
Wesley moving away from James at 7.4 ms^{-1} ✓
along 90 + 170.6 = 260.6°. ✓

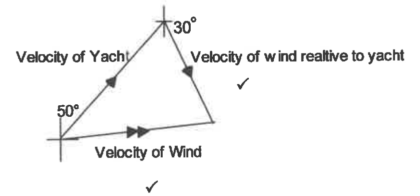
```
toPol([-7.2943,-1.2139]
   [7.3946 ∠(-170.5516)]
```

Calculator Assumed

2. [6 marks: 2, 4]

A yacht is sailing on a bearing of 050° with speed 12 kmh^{-1}. Sarah on the yacht measures the wind as blowing with a speed of 10 kmh^{-1} from a bearing of 300°.

(a) Sketch a clearly labelled velocity vector diagram that shows the relationship between the velocity of the yacht, the true velocity of the wind and the velocity of wind relative to the yacht.

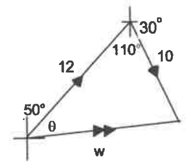


(b) Find the true speed and direction of the wind.

In the accompanying diagram:

$w^2 = 12^2 + 10^2 - 2(12)(10)\cos 110$

$w = 18.0578$ ✓



Also: $\dfrac{\sin\theta}{10} = \dfrac{\sin 110}{18.0578}$

$\Rightarrow \sin\theta = 0.5204$

$\theta = 31.36°$ ✓

Hence, wind is blowing with a speed of 18.1 kmh^{-1} along bearing $(050 + 31.36)° = 081.4°$. ✓✓

OR

${}_w v_y = <10\cos 30, -10\sin 30>$, $v_y = <12\sin 50, 12\cos 50>$

$v_w = {}_w v_y + v_y$

$= <10\cos 30 + 12\sin 50, -10\sin 30 + 12\cos 50>$ ✓

$= <17.8528, 2.7135>$ ✓

$= \text{Polar}[18.0578, 8.6423°]$

Hence, wind is blowing with a speed of 18.1 kmh^{-1} along bearing $(050 + 31.36)° = 081.4°$. ✓✓

Calculator Assumed

3. [7 marks: 2, 2, 3]

A yacht Y is moving with velocity **2i** + 5**j** kmh^{-1}. A sailor on board the yacht measures the wind as blowing with velocity –3**i** – 2**j** kmh^{-1}.

(a) Find the velocity of the wind.

$v_y = <2, 5>$

${}_w v_y = <-3, -2> \quad \Rightarrow v_w - v_y = <-3, -2>$

Hence, $v_w = <2, 5> + <-3, -2>$ ✓

$v_w = <-1, 3>$ ✓

To a sailor on a second yacht Z, the wind appears to be blowing with velocity 2**i** + 4**j** kmh^{-1}.

(b) Find the velocity of the second yacht.

$v_w = <-1, 3>$

${}_w v_z = <2, 4> \quad \Rightarrow v_w - v_z = <2, 4>$

Hence, $v_z = <-1, 3> - <2, 4>$ ✓

$v_z = <-3, -1>$ ✓

(c) How fast is yacht Z moving away from yacht Y and in what direction?

${}_z v_y = <-3, -1> - <2, 5>$

$= <-5, -6>$ ✓

$= \text{Polar}[7.8102, -129.8056°]$

Z is moving away with a speed of 7.8 kmh^{-1} ✓

along bearing 219.8°. ✓

```
toPol([-5,-6]
    [7.8102 ∠(-129.8056)]
```

Calculator Assumed

4. [4 marks]

A ship is travelling with a speed of 20 knots along bearing 080°. Relative to the ship, the wind is blowing from 310° with a speed of 8 knots. Let **i** and **j** be unit vectors in the Easterly and Northerly directions respectively. By expressing the given velocities in component form, find the true speed and direction of the wind.

> Velocity of Ship $v_s = 20\sin 80\,\mathbf{i} + 20\cos 80\,\mathbf{j}$
> Velocity of Wind relative to the ship ${}_w v_s = 8\sin 50\,\mathbf{i} - 8\cos 50\,\mathbf{j}$
>
> ${}_w v_s = \mathbf{v}_w - \mathbf{v}_s$
>
> $\Rightarrow v_w = (8\sin 50\,\mathbf{i} - 8\cos 50\,\mathbf{j}) + (20\sin 80\,\mathbf{i} + 20\cos 80\,\mathbf{j})$
>
> $= 25.8245\,\mathbf{i} - 1.6693\,\mathbf{j}$ ✓✓
>
> $=$ Polar$[25.8784, -3.698^\circ]$
>
> Hence, true speed of the wind is 25.9 knots along bearing $(90 + 3.6984) = 093.7^\circ$. ✓✓
>
> ```
> toPol([25.8245,-1.6693]
>       [25.8784 ∠(-3.6985)]
> ```


Calculator Assumed

5. [11 marks: 2, 2, 7]

May is running along 025° at 4 kmh^{-1}.
Relative to May, Fay is running with speed a kmh^{-1} along bearing 150°.

(a) Find in terms of a, the velocity of Fay, v_f.

> ${}_f v_m = \langle a\sin 30, -a\cos 30\rangle,\ v_m = \langle 4\sin 25, 4\cos 25\rangle$
>
> $v_f = {}_f v_m + v_m$
>
> $= \langle a\sin 30 + 4\sin 25, -a\cos 30 + 4\cos 25\rangle$. ✓✓

Jane is running due East at 2 kmh^{-1}.
Relative to Jane, Fay is running with speed b kmh^{-1} along bearing 120°.

(b) Find in terms of b, the velocity of Fay, v_f.

> ${}_f v_j = \langle b\sin 60, -b\cos 60\rangle,\ v_j = \langle 2, 0\rangle$
>
> $v_f = {}_f v_j + v_j$
>
> $= \langle b\sin 60 + 2, -b\cos 60\rangle$. ✓✓

(c) Use your answers in (a) & (b) to find a and b. Hence, find the true speed and direction with which Fay is running.

> From (a), $v_f = \langle a\sin 30 + 4\sin 25, -a\cos 30 + 4\cos 25\rangle$. I
>
> From (b), $v_f = \langle b\sin 60 + 2, -b\cos 60\rangle$. II
>
> Compare I with II:
> $a\sin 30 + 4\sin 25 = b\sin 60 + 2$ III ✓
> $-a\cos 30 + 4\cos 25 = -b\cos 60$ IV ✓
>
> Solve III & IV simultaneously:
> $\Rightarrow a = 5.9696,\ b = 3.0891$ ✓✓
>
> Hence, $v_f = \langle 3.0891\sin 60 + 2, -3.0891\cos 60\rangle$
> $= \langle 4.6752, -1.5446\rangle$ ✓
>
> Hence, true speed for Fay = 4.9 kmh^{-1} ✓
> Direction = 90 + 18.3 = 108.3° ✓
>
> ```
> { a×sin(30)+4×sin(25)=b×sin(60)+2
> { -a×cos(30)+4×cos(25)=-b×cos(60) | a,b
>                               {a=5.9696,b=3.0891}
> [b×sin(60)+2,-b×cos(60)]|b=3.08911
>                                 [4.6752  -1.5446]
> toPol(
>                              [4.9238 ∠(-18.2819)]
> ```

Calculator Assumed

6. [9 marks: 3, 3, 3]

Tom is in a hot-air balloon travelling with velocity 3**i** + 4**j** km and to Tom the wind is blowing along bearing 045°. Jerry is in another hot-air balloon flying at the same altitude and travelling with velocity –4**i** + **j**. To Jerry, the wind is blowing along bearing 60°. Let the true velocity of the wind be $x\mathbf{i} + y\mathbf{j}$ kmh^{-1}. **i** and **j** are unit vectors in the Easterly and Northerly directions respectively.

(a) Find in terms of x and y, the velocity of the wind relative to Tom. Hence, show that $x - y = -1$.

> $_wv_t = \langle x, y\rangle - \langle 3, 4\rangle$
> $= \langle x-3, y-4\rangle$ ✓
> Since, the velocity of wind relative to Tom is along 045°,
> $\dfrac{y-4}{x-3} = \tan 45 = 1$ ✓
> $\Rightarrow\ y - 4 = x - 3 \ \Rightarrow\ x - y = -1$ ✓

(b) Find in terms of x and y, the velocity of the wind relative to Jerry. Hence, show that $x - \sqrt{3}y = -4 - \sqrt{3}$.

> $_wv_j = \langle x, y\rangle - \langle -4, 1\rangle$
> $= \langle x+4, y-1\rangle$ ✓
> Since, the velocity of wind relative to Tom is along 060°,
> $\dfrac{y-1}{x+4} = \tan(90-60) = \dfrac{1}{\sqrt{3}}$ ✓
> $\Rightarrow\ \sqrt{3}y - \sqrt{3} = x + 4 \ \Rightarrow\ x - \sqrt{3}y = -4 - \sqrt{3}$ ✓

(c) Find the true velocity of the wind.

> Solve simultaneously:
> $x - y = -1$
> $x - \sqrt{3}y = -4 - \sqrt{3}$
> $\Rightarrow x = 5.46, y = 6.46$ ✓✓
>
> Hence, true velocity of wind = (5.46**i** + 6.46**j**). ✓

Calculator Assumed

7. [4 marks: 1, 2, 2]

At 1000 hours, the position vectors of A and B are $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. A is moving with velocity $\boldsymbol{u}$ and B is moving with velocity $\boldsymbol{v}$.

(a) Find $_B\boldsymbol{r}_A$ the position vector of B relative to A.

> $_B\boldsymbol{r}_A = \boldsymbol{b} - \boldsymbol{a}$ ✓

(b) Find $_A\boldsymbol{v}_B$ the velocity of A relative to B.

> $_A\boldsymbol{v}_B = \boldsymbol{u} - \boldsymbol{v}$ ✓

(c) Show that if A collides with B after t seconds, $_B\boldsymbol{r}_A = t\ _A\boldsymbol{v}_B$ where $t > 0$.

> For A to collide with B after t seconds:
> $\boldsymbol{a} + t\boldsymbol{u} = \boldsymbol{b} + t\boldsymbol{v}$ ✓
> $\Rightarrow\ \boldsymbol{b} - \boldsymbol{a} = t(\boldsymbol{u} - \boldsymbol{v})$ ✓
> $_B\boldsymbol{r}_A = t \times {_A\boldsymbol{v}_B}$

8. [4 marks: 3, 1]

At 0800 hours, the position vector of Q relative to P is 45**i** – 60**j** km. The velocity of P relative to Q is a constant 9**i** + m **j** kmh^{-1}.

(a) Find the value of m, if P intercepts Q.

> $_Q\boldsymbol{r}_P = 45\mathbf{i} - 60\mathbf{j}$ and $_P\boldsymbol{v}_Q = 9\mathbf{i} + m\mathbf{j}$
> If P intercepts Q, then $_Q\boldsymbol{r}_P = t\ _P\boldsymbol{v}_Q$.
> Hence, $45\mathbf{i} - 60\mathbf{j} = t(9\mathbf{i} + m\mathbf{j})$ ✓
> $\Rightarrow\ t = 5$ ✓
> $m = -12$ ✓

(b) Find when P intercepts Q.

> P intercepts Q after 5 hours (λ = 5), ie at 1300 hours. ✓

Calculator Assumed

9. [6 marks: 1, 1, 4]

At 0900 hours, P is located at $20\mathbf{i} - 10\mathbf{j}$ km and is travelling with a constant velocity of $-4\mathbf{i} + 6\mathbf{j}$ kmh^{-1}. At the same time, Q is located at $50\mathbf{i} + 40\mathbf{j}$ km and is travelling with a constant velocity of $-10\mathbf{i} - 4\mathbf{j}$ kmh^{-1}.

(a) Find the displacement of Q relative to P at 0900 hours.

$${}_Q r_P = r_Q - r_P = (50\mathbf{i} + 40\mathbf{j}) - (20\mathbf{i} - 10\mathbf{j}) = 30\mathbf{i} + 50\mathbf{j} \quad ✓$$

(b) Find the velocity of P relative to Q at 0900 hours.

$${}_P v_Q = v_P - v_Q = (-4\mathbf{i} + 6\mathbf{j}) - (-10\mathbf{i} - 4\mathbf{j}) = 6\mathbf{i} + 10\mathbf{j} \quad ✓$$

(c) Use your answers in (a) and (b) to determine when and where P will collide with Q.

For P to collide with Q, ${}_Q r_P = t\,{}_P v_Q$.

Hence, $30\mathbf{i} + 50\mathbf{j} = t(6\mathbf{i} + 10\mathbf{j})$ ✓

$\Rightarrow t = 5$ ✓

P will collide with Q after 5 hours ($t = 5$), ie at 1400 hours. ✓
The collision will take place at $(20\mathbf{i} - 10\mathbf{j}) + 5(-4\mathbf{i} + 6\mathbf{j}) = 20\mathbf{j}$. ✓

Calculator Assumed

10. [11 marks: 2, 2, 5, 2]

When the clock struck one, relative to a clock tower, the position vector of a cat is $6\mathbf{i} + 10\mathbf{j}$ m. The cat is running with velocity $2\mathbf{i} + 2\mathbf{j}$ ms^{-1}. At the same time, relative to the same clock tower, the position vector of a dog is $-4\mathbf{i} - 6\mathbf{j}$. The dog can run at 5 ms^{-1}.

(a) Find the position vector of the cat relative to the dog.

${}_C r_D = r_C - r_D$ [where C: cat, D: dog]
$= (6\mathbf{i} + 10\mathbf{j}) - (-4\mathbf{i} - 6\mathbf{j}) = 10\mathbf{i} + 16\mathbf{j}$ ✓

Let the velocity vector of the dog for it to intercept the cat be $x\mathbf{i} + y\mathbf{j}$ ms^{-1}.

(b) Find in terms of x and y, the velocity of the dog relative to the cat.

$${}_D v_C = v_D - v_C = (x\mathbf{i} + y\mathbf{j}) - (2\mathbf{i} - 2\mathbf{j}) = (x-2)\mathbf{i} + (y-2)\mathbf{j} \quad ✓$$

(c) Use your answers in (a) and (b) to find x and y for the dog to intercept the cat. Assume that the cat continues running with the same speed and in the same direction.

For the dog to intercept the cat, ${}_C r_D = t\,{}_D v_C$ where $t > 0$.

Hence, $10\mathbf{i} + 16\mathbf{j} = t[(x-2)\mathbf{i} + (y-2)\mathbf{j}]$

$\Rightarrow t(x-2) = 10$ I ✓

$t(y-2) = 16$ II ✓

But, speed of dog is 5.

Hence, $x^2 + y^2 = 25$ III ✓

Solve I, II and III simultaneously:

$x = 3.1678$, $y = 3.8685$, $t = 8.5631$ ✓✓

```
t(x-2)=10
t(y-2)=16
x^2+y^2=5^2 | x,y,t
{{x=-2.0891,y=-4.5426,t=-2.4455},{x=3.1678,y=3.8685,t=8.5631}}
```

(d) When and where does the interception take place.

Hence, interception will occur after 8.6 seconds at ✓
$(6\mathbf{i} + 10\mathbf{j}) + 8.56(2\mathbf{i} + 2\mathbf{j}) = 23.12\mathbf{i} + 27.12\mathbf{j}$. ✓

Calculator Assumed

11. [16 marks: 1, 1, 3, 1, 4, 3, 3]

The locations and velocities of three boats at 0900 hours are given in the table below. Assume that each boat moves with constant velocity.

Boat	Location (km)	Velocity (kmh^{-1})
A	20i + 40j	−4i − 6j
B	−25i + 20j	5i − 2j
C	35i − 10j	−7i + 4j

(a) Find the position vector of A relative to B at 0900 hours.

$_A r_B$ = (20i + 40j) − (−25i + 20j)
= 45i + 20j ✓

(b) Find the velocity of B relative to A.

$_B v_A$ = (5i − 2j) − (−4i − 6j)
= 9i + 4j ✓

(c) Use your answer in (b) to determine at what time B will meet A.

When B meets A:
$_A r_B = t\ _B v_A$
45i + 20j = t (9i + 4j) ✓
⇒ $t = 5$ ✓

Hence, B will meet A 5 hours after 0900 hours, ie at 1400 hours. ✓

(d) Determine the position vector of the point where A will meet B.

Position vector of meeting point = (20i + 40j) + 5 (−4i − 6j)
= 10j ✓

Calculator Assumed

11. (e) Find the distance between A and C t hours after 0900 hours.

OA = (20i + 40j) + t (−4i − 6j)
= (20 − 4t) i + (40 − 6t) j ✓

OC = (35i − 10j) + t (−7i + 4j)
= (35 − 7t) i + (−10 + 4t) j ✓

AC = [(35 − 7t) i + (−10 + 4t) j] − [(20 − 4t) i + (40 − 6t) j]
= (15 − 3t) i + (−50 + 10t) j ✓

Hence, $|\mathbf{AC}| = \sqrt{[(15-3t)^2 + (-50+10t)^2]}$
$= \sqrt{109(t-5)^2}$ ✓

(f) Find the time when C will meet A.

When C meets A, $|\mathbf{AC}| = 0$ ✓
⇒ $\sqrt{109(t-5)^2} = 0$
⇒ $t = 5$ ✓
Hence, C will meet A at 1400 hours. ✓

(g) Prove that all three boats will arrive at a similar spot at the same time.

Since B will meet A at 10j at 1400 hours and
C will meet A at 1400 hours, A, B and C will meet at the same spot (10j) at 1400 hours. ✓✓✓

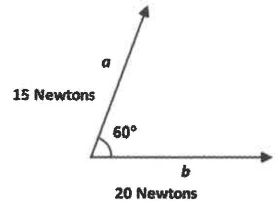
12 Scalar Product I

Calculator Free

1. [4 marks: 2, 2]

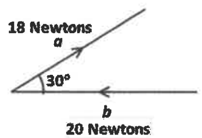
Find the scalar product between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as given:

(a)



Scalar product = $15 \times 20 \times \cos 60$ ✓
= 150 ✓

(b)



Scalar product = $20 \times 18 \times \cos(180 - 30)$ ✓
= $-180\sqrt{3}$ ✓

---

2. [6 marks: 2, 2, 2]

Given that vector $\boldsymbol{u}$ has magnitude 10 ms^{-1} in the direction 030°, $\boldsymbol{v}$ has magnitude 15 ms^{-1} in the direction 090° and $\boldsymbol{w}$ has magnitude 5 ms^{-1} in the direction 180°.
Find:

(a) $\boldsymbol{u} \cdot \boldsymbol{v}$

$\boldsymbol{u} \cdot \boldsymbol{v} = 10 \times 15 \times \cos 60$
= 75 ✓✓

(b) $\boldsymbol{u} \cdot \boldsymbol{w}$

$\boldsymbol{u} \cdot \boldsymbol{w} = 10 \times 5 \times \cos 150$
= $-25\sqrt{3}$ ✓✓

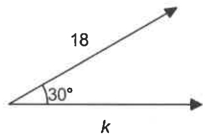
(c) the magnitude and direction of $(\boldsymbol{u} \cdot \boldsymbol{w})\,\boldsymbol{v}$.

Magnitude = $|-25\sqrt{3}| \times 15 = 375\sqrt{3}$ ✓
Direction = opposite to that of $\boldsymbol{v}$ = bearing 270° ✓

Calculator Free

3. [2 marks]

The scalar product between the two vectors shown is $180\sqrt{3}$. Find k.

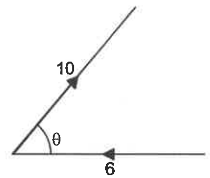


$18 \times k \times \cos 30 = 180\sqrt{3}$ ✓
$\Rightarrow k = 20$ ✓

---

4. [2 marks]

The scalar product between the two vectors shown is –50. Find cos θ.



$10 \times 6 \times \cos(180 - \theta) = -50$ ✓
$-60 \cos\theta = -50$
$\cos\theta = \frac{5}{6}$ ✓

---

5. [5 marks]

Given that $|\boldsymbol{a}| = 20$ and $|\boldsymbol{b}| = 25$, find with reasons the maximum and minimum value of $\boldsymbol{a} \cdot \boldsymbol{b}$.

$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$ (where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$)
$= 20 \times 25 \times \cos\theta$
$= 500 \cos\theta$ ✓

But maximum value for $\cos\theta = 1$. ✓
$\Rightarrow$ Maximum value for $\boldsymbol{a} \cdot \boldsymbol{b} = 500$. ✓

Also minimum value for $\cos\theta = -1$. ✓
$\Rightarrow$ Minimum value for $\boldsymbol{a} \cdot \boldsymbol{b} = -500$. ✓

Calculator Free

6. [4 marks: 2, 1, 1]

Given that $|\boldsymbol{a}| = 8$ and $|\boldsymbol{b}| = 5$, and if $\boldsymbol{a} \cdot \boldsymbol{b} = 20\sqrt{2}$, find θ, the acute angle between:

(a) $\boldsymbol{a}$ and $\boldsymbol{b}$

$\cos\theta = \dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \dfrac{20\sqrt{2}}{8\times 5}$ ✓

$= \dfrac{\sqrt{2}}{2} \Rightarrow \theta = 45^\circ$ ✓

(b) $2\boldsymbol{a}$ and $3\boldsymbol{b}$

Angle = 45° ✓

(c) $\boldsymbol{a}$ and $-\boldsymbol{b}$.

Angle = 45° ✓

7. [6 marks: 3, 3]

Given that $|\boldsymbol{m}| = 10$ and $|\boldsymbol{n}| = 10$, find $\boldsymbol{n}$ in terms of $\boldsymbol{m}$ if:

(a) $\boldsymbol{m} \cdot \boldsymbol{n} = 100$

Let the angle between $\boldsymbol{m}$ and $\boldsymbol{n}$ be θ.

$\Rightarrow \cos\theta = \dfrac{\boldsymbol{m}.\boldsymbol{n}}{|\boldsymbol{m}||\boldsymbol{n}|} = \dfrac{100}{10\times 10}$ ✓

$\Rightarrow \cos\theta = 1$

$\Rightarrow \theta = 0^\circ$ ✓

Since $\boldsymbol{m}$ and $\boldsymbol{n}$ have the same magnitude and are in the same direction, $\boldsymbol{n} = \boldsymbol{m}$. ✓

(b) $\boldsymbol{m} \cdot \boldsymbol{n} = -100$

Let the angle between $\boldsymbol{m}$ and $\boldsymbol{n}$ be θ.

$\Rightarrow \cos\theta = \dfrac{\boldsymbol{m}.\boldsymbol{n}}{|\boldsymbol{m}||\boldsymbol{n}|} = \dfrac{-100}{10\times 10}$ ✓

$\Rightarrow \cos\theta = -1$

$\Rightarrow \theta = 180^\circ$ ✓

Since $\boldsymbol{m}$ and $\boldsymbol{n}$ have the same magnitude and are in opposite directions, $\boldsymbol{n} = -\boldsymbol{m}$. ✓

Calculator Free

8. [4 marks: 1, 1, 2]

Given that $\boldsymbol{a} = \begin{pmatrix}1\\2\end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix}-1\\3\end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix}1\\2\end{pmatrix}$, find:

(a) $(\boldsymbol{a}+\boldsymbol{b}) \cdot \boldsymbol{c}$

$\left[\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}-1\\3\end{pmatrix}\right]\cdot\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\5\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix}$

$= 10$ ✓

(b) $(\boldsymbol{a}+\boldsymbol{b}) \cdot (\boldsymbol{a}+\boldsymbol{c})$

$\left[\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}-1\\3\end{pmatrix}\right]\cdot\left[\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}\right] = \begin{pmatrix}0\\5\end{pmatrix}\cdot\begin{pmatrix}2\\4\end{pmatrix}$

$= 20$ ✓

(c) k if $\boldsymbol{a} \cdot (3\mathbf{i} + k\mathbf{j}) = 9$

$(\mathbf{i} + 2\mathbf{j}) \cdot (3\mathbf{i} + k\mathbf{j}) = 9$ ✓

$\Rightarrow 3 + 2k = 9$

$k = 3$ ✓

9. [3 marks: 1, 1, 1]

Expand and simplify:

(a) $(\boldsymbol{m}+\boldsymbol{n}) \cdot (\boldsymbol{m}+\boldsymbol{n})$

$(\boldsymbol{m}+\boldsymbol{n}) \cdot (\boldsymbol{m}+\boldsymbol{n}) = \boldsymbol{m}\cdot\boldsymbol{m} + \boldsymbol{m}\cdot\boldsymbol{n} + \boldsymbol{n}\cdot\boldsymbol{m} + \boldsymbol{n}\cdot\boldsymbol{n}$

$= |\boldsymbol{m}|^2 + 2\,\boldsymbol{m}\cdot\boldsymbol{n} + |\boldsymbol{n}|^2$ ✓

(b) $(\boldsymbol{c}+\boldsymbol{d}) \cdot (\boldsymbol{c}-\boldsymbol{d})$

$(\boldsymbol{c}+\boldsymbol{d}) \cdot (\boldsymbol{c}-\boldsymbol{d}) = \boldsymbol{c}\cdot\boldsymbol{c} - \boldsymbol{c}\cdot\boldsymbol{d} + \boldsymbol{d}\cdot\boldsymbol{c} - \boldsymbol{d}\cdot\boldsymbol{d}$

$= |\boldsymbol{c}|^2 - |\boldsymbol{d}|^2$ ✓

(c) $(\boldsymbol{m}-3\boldsymbol{n}) \cdot (2\boldsymbol{m}-\boldsymbol{n})$

$(\boldsymbol{m}-3\boldsymbol{n}) \cdot (2\boldsymbol{m}-\boldsymbol{n}) = \boldsymbol{m}\cdot 2\boldsymbol{m} - \boldsymbol{m}\cdot\boldsymbol{n} - 3\boldsymbol{n}\cdot 2\boldsymbol{m} + 3\boldsymbol{n}\cdot\boldsymbol{n}$

$= 2|\boldsymbol{m}|^2 - 7\,\boldsymbol{m}\cdot\boldsymbol{n} + 3|\boldsymbol{n}|^2$ ✓

Calculator Free

10. [4 marks: 2, 2]

Given that p and q are perpendicular, prove that:

(a) $p \cdot q = 0$

$p \cdot q = |p|\,|q| \cos 90$
$= 0$ ✓✓

(b) $(p + q) \cdot (p + q) = |p|^2 + |q|^2$

$(p+q)\cdot(p+q) = p \cdot p + p \cdot q + q \cdot p + q \cdot q$
$= |p|^2 + 2p \cdot q + |q|^2$ ✓
$= |p|^2 + |q|^2$ since $p \cdot q = 0$ ✓

---

11. [7 marks: 2, 2, 3]

Given that $|r| = 10$ and $|s| = 8$, find:

(a) $r \cdot r$

$r \cdot r = |r|\,|r| \cos 0$
$= 10 \times 10 \times \cos 0$ ✓
$= 100$ ✓

(b) $(r + s) \cdot (r + s)$ if r and s are parallel and in the same direction.

$(r+s)\cdot(r+s) = |r|^2 + 2r \cdot s + |s|^2$
$= 10^2 + 2[|r|\,|s| \cos 0] + 8^2$ ✓
$= 100 + (2 \times 10 \times 8) + 64$
$= 324$ ✓

(c) $|r - s|$ if r and s are perpendicular.

$|r-s|^2 = (r-s)\cdot(r-s)$
$= |r|^2 - 2r \cdot s + |s|^2$ ✓
$= 10^2 - 2 \times 0 + 8^2$ ✓ [r and s perpendicular $\Rightarrow r \cdot s = 0$]
$= 164$

$|r - s| = 2\sqrt{41}$ ✓

Calculator Assumed

12. [7 marks: 2, 3, 2]

Given that $u = i + 3j$, $v = -6i + 8j$ and $w = ki - 2j$, find:

(a) the acute angle between u and v.

$\cos\theta = \dfrac{u \cdot v}{|u|\,|v|} = \dfrac{\begin{pmatrix}1\\3\end{pmatrix}\begin{pmatrix}-6\\8\end{pmatrix}}{\sqrt{10} \times 10}$ ✓ [θ is the angle between u and v]

$= \dfrac{18}{10\sqrt{10}} \Rightarrow \theta = 55.30^\circ$ ✓

```
angle([1 3],[-6 8])
                 55.3048
```

(b) k, if v and w are parallel in the opposite direction.

$\Rightarrow v = -\lambda w$ where λ is a positive constant
$-6i + 8j = -\lambda(ki - 2j)$ ✓
Compare i and j coefficients:
$-6 = -\lambda k$
$8 = 2\lambda \Rightarrow \lambda = 4$ ✓
Hence, $k = \dfrac{3}{2}$ ✓

(c) k, if the angle between v and w is 120°.

$\cos 120 = \dfrac{\begin{pmatrix}-6\\8\end{pmatrix}\begin{pmatrix}k\\-2\end{pmatrix}}{10 \times \sqrt{(k^2+4)}}$ ✓

$\Rightarrow -\dfrac{1}{2} = \dfrac{\begin{pmatrix}-6\\8\end{pmatrix}\begin{pmatrix}k\\-2\end{pmatrix}}{10 \times \sqrt{(k^2+4)}}$

$-6k - 16 = -5\sqrt{(k^2+4)}$
$\Rightarrow k = -0.8543$ ✓

```
solve(-6x-16=-5√(x²+4) ,x)
                 {x=-0.8543}
```

OR
Use CAS:
solve(angle(< −6, 8 >, < k, −2 >) = 120, k) ✓
$\Rightarrow k = -0.8543$ ✓

```
solve(angle([-6 8],[k -2])=120,k)
                 {k=-0.8543145111}
```

Calculator Assumed

13. [6 marks: 3, 3]

Given that $|\boldsymbol{u}| = 10$ and $(\boldsymbol{u} + 2\boldsymbol{v}) \,.\, (\boldsymbol{u} + \boldsymbol{v}) = 408$.

(a) Find $|\boldsymbol{v}|$ if $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular.

$(\boldsymbol{u} + 2\boldsymbol{v}) \,.\, (\boldsymbol{u} + \boldsymbol{v}) = |\boldsymbol{u}|^2 + 3\,\boldsymbol{u}\,.\,\boldsymbol{v} + 2|\boldsymbol{v}|^2$ ✓

$\Rightarrow \quad 408 = 100 + (3 \times 0) + 2|\boldsymbol{v}|^2$ ✓

$\Rightarrow 2|\boldsymbol{v}|^2 = 308$

$|\boldsymbol{v}| = \sqrt{154}$ ✓

(b) Find $|\boldsymbol{v}|$ if $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel and in opposite directions.

$(\boldsymbol{u} + 2\boldsymbol{v}) \,.\, (\boldsymbol{u} + \boldsymbol{v}) = |\boldsymbol{u}|^2 + 3\,\boldsymbol{u}\,.\,\boldsymbol{v} + 2|\boldsymbol{v}|^2$

$\Rightarrow \quad 408 = 100 + (3 \times 10 \times |\boldsymbol{v}| \times \cos 180) + 2|\boldsymbol{v}|^2$ ✓

$\Rightarrow 2|\boldsymbol{v}|^2 - 30|\boldsymbol{v}| - 308 = 0$ ✓

$|\boldsymbol{v}| = 22$ or -7 (reject as $|\boldsymbol{v}| > 0$)

Hence, $|\boldsymbol{v}| = 22$. ✓

---

14. [6 marks: 3, 3]

Given that $\boldsymbol{u} = -\mathbf{i} + \mathbf{j}$, find in exact form, a unit vector $\hat{\boldsymbol{v}}$, if:

(a) $\boldsymbol{u}$ is perpendicular to $\hat{\boldsymbol{v}}$.

A vector that is perpendicular to $\boldsymbol{u}$ is $\mathbf{i} + \mathbf{j}$. ✓

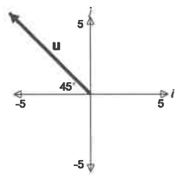
$|\mathbf{i} + \mathbf{j}| = \sqrt{2}$

Hence, $\hat{\boldsymbol{v}} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$

$= \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$ ✓✓

[other equivalent answers possible]

(b) the acute angle between $\boldsymbol{u}$ and $\hat{\boldsymbol{v}}$ is 45°.



From sketch that $\hat{\boldsymbol{v}} = -\mathbf{i}$ or $\mathbf{j}$. ✓✓✓

Calculator Assumed

15. [6 marks]

Given that $\boldsymbol{u} = 3\mathbf{i} + \mathbf{j}$ and $\boldsymbol{v} = x\mathbf{i} + y\mathbf{j}$, find x and y if $|\boldsymbol{v}| = \sqrt{10}$ and the acute angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is 60°.

$$\cos 60 = \frac{\begin{pmatrix}3\\1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}}{\sqrt{10} \times \sqrt{10}}$$ ✓✓

$\frac{1}{2} = \frac{3x + y}{10}$ where $3x + y > 0$

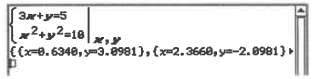
$\Rightarrow 3x + y = 5$ I ✓

$|\boldsymbol{v}| = \sqrt{10} \quad \Rightarrow x^2 + y^2 = 10$ II ✓

Solve I & II simultaneously:

Hence, $x = 0.6340, y = 3.0981$ ✓

or $x = 2.3660, y = -2.0981$ ✓



13 Scalar Product II

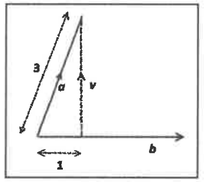
Calculator Free

1. [9 marks: 1, 2, 2, 2, 2]

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have magnitudes 3 and 4 respectively. The acute angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\cos^{-1}\frac{1}{3}$. Find in terms of the vectors $\boldsymbol{a}$ and/or $\boldsymbol{b}$:

(a) the scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

Scalar projection of $\boldsymbol{a}$ onto $\boldsymbol{b} = 3 \times \frac{1}{3} = 1$. ✓



(b) the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

$\text{proj}_b\, \boldsymbol{a} = 1 \times \hat{\boldsymbol{b}}$ ✓

$= 1 \times \frac{1}{4}\boldsymbol{b} = \frac{1}{4}\boldsymbol{b}$ ✓

(c) the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$.

$\text{proj}_a\, \boldsymbol{b} = 4 \times \frac{1}{3} \times \hat{\boldsymbol{a}}$ ✓

$= \frac{4}{3} \times \frac{1}{3}\boldsymbol{a} = \frac{4}{9}\boldsymbol{a}$ ✓

(d) $\boldsymbol{v}$, the component of $\boldsymbol{a}$ that is perpendicular to $\boldsymbol{b}$.

$\boldsymbol{v} + \text{proj}_b\, \boldsymbol{a} = \boldsymbol{a}$

$\boldsymbol{v} = \boldsymbol{a} - \frac{1}{4}\boldsymbol{b}$ ✓✓

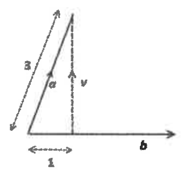
(e) the magnitude of the $\boldsymbol{v}$, the component of $\boldsymbol{a}$ that is perpendicular to $\boldsymbol{b}$.

Using Pythagoras Theorem:

$|\boldsymbol{v}|^2 = 3^2 - |\text{proj}_b\, \boldsymbol{a}|^2$ ✓

$= 9 - 1$

$|\boldsymbol{v}| = 2\sqrt{2}$ ✓



Calculator Free

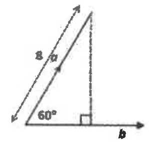
2. [10 marks: 2, 2, 3, 3]

(a) The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have magnitudes 8 and 6 respectively. Find the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ if the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

(i) is 60°.

$\text{proj}_b\, \boldsymbol{a} = 8\cos 60 \times \hat{\boldsymbol{b}}$ ✓

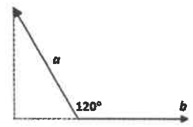
$= 4 \times \frac{1}{6}\boldsymbol{b} = \frac{2}{3}\boldsymbol{b}$ ✓



(ii) is 120°.

$\text{proj}_b\, \boldsymbol{a} = 8\cos 120 \times \hat{\boldsymbol{b}}$ ✓

$= -4 \times \frac{1}{6}\boldsymbol{b} = -\frac{2}{3}\boldsymbol{b}$ ✓



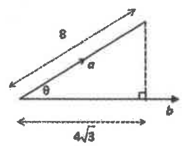
(b) Find θ, the acute angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ if $|\boldsymbol{a}| = 8$ and the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ has a magnitude of $4\sqrt{3}$.

$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{\boldsymbol{a}\cdot\hat{\boldsymbol{b}}}{|\boldsymbol{a}|}$

But $\boldsymbol{a}\cdot\hat{\boldsymbol{b}} = 4\sqrt{3}$

Hence, $\cos\theta = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$ ✓✓

$\theta = 30^\circ$ ✓



(c) The vector $\boldsymbol{b}$ has magnitude 10. The projection of vector $\boldsymbol{a}$ on $\boldsymbol{b}$ is $5\boldsymbol{b}$. Find the magnitude of $\boldsymbol{a}$ if the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 60°.

$|\text{proj}_b\, \boldsymbol{a}| = |5\boldsymbol{b}|$

$= 5 \times 10 = 50$ ✓

But $|\text{proj}_b\, \boldsymbol{a}| = |\boldsymbol{a}|\cos 60$.

$\Rightarrow |\boldsymbol{a}|\cos 60 = 50$ ✓

$|\boldsymbol{a}| = 100$. ✓

Calculator Free

3. [4 marks: 2, 2]

Given $\boldsymbol{a} = <5, 10>$ and $\boldsymbol{b} = <4, 3>$.

(a) Find the component of $\boldsymbol{a}$ that is parallel to $\boldsymbol{b}$.

$\text{proj}_b\, \boldsymbol{a} = (\boldsymbol{a} \bullet \hat{\boldsymbol{b}})\hat{\boldsymbol{b}}$

$= [<5, 10> \bullet \frac{1}{5}<4, 3>]\frac{1}{5}<4, 3>$ ✓

$= 10 \times \frac{1}{5}<4, 3>$

$= <8, 6>$ ✓

(b) Find the component of $\boldsymbol{a}$ that is perpendicular to $\boldsymbol{b}$.

Let required vector be $\boldsymbol{v}$.
Hence, $\boldsymbol{v} + \text{proj}_b\, \boldsymbol{a} = \boldsymbol{a}$

$\boldsymbol{v} = <5, 10> - <8, 6>$ ✓

$= <-3, 4>$ ✓

4. [6 marks: 2, 2, 2]

Given that $\boldsymbol{a} = <2, 5> + <10, -4>$, find:

(a) the vector projection of $\boldsymbol{a}$ onto $<6, 15>$.

$<2, 5>$ and $<10, -4>$ are components of $\boldsymbol{a}$
parallel and perpendicular to $<6, 15>$ respectively. ✓
Hence, vector projection of $\boldsymbol{a}$ onto $<6, 15> = <2, 5>$. ✓

(b) the vector projection of $\boldsymbol{a}$ onto $<-5, 2>$

$<10, -4>$ and $<2, 5>$ are components of $\boldsymbol{a}$
parallel and perpendicular to $<-5, 2>$ respectively. ✓
Hence, vector projection of $\boldsymbol{a}$ onto $<-5, 2> = <10, -4>$. ✓

(c) the vector projection of $\boldsymbol{a}$ onto $<-5, 0>$

$\boldsymbol{a} = <12, -1> = <12, 0> + <0, -1>$.
$<12, 0>$ and $<0, -1>$ are components of $\boldsymbol{a}$
parallel and perpendicular to $<-5, 0>$. ✓
Hence, vector projection of $\boldsymbol{a}$ onto $<-5, 0>$ is $<12, 0>$. ✓

Calculator Free

5. [11 marks: 3, 2, 2, 4]

The acute angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is 60°. The vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is $<4, -3>$ and $|\boldsymbol{v}| = 10$.

(a) Find $\boldsymbol{v}$.

$\text{proj}_v\, \boldsymbol{u} = <4, -3>$
Hence, $\boldsymbol{v} = \mu <4, -3>$. ✓
But $|\boldsymbol{v}| = 10$.

$|\mu <4, -3>| = 10$ ✓

$\mu = 2$.

Hence, $\boldsymbol{v} = <8, -6>$. ✓

(b) Explain clearly why $\boldsymbol{u} = <4, -3> + \lambda <3, 4>$.

$<4, -3>$ is the vector component of $\boldsymbol{u}$ parallel to $<8, -6>$. ✓
$\lambda <3, 4>$ is perpendicular to $<8, -6>$ as $\lambda <3, 4> \bullet <8, 6> = 0$.
Hence, $\lambda <3, 4>$ is the vector component of $\boldsymbol{u}$ perpendicular to $<8, -6>$. ✓
Hence, $\boldsymbol{u} = <4, -3> + \lambda <3, 4>$.

(c) Find $|\boldsymbol{u}|$.

$\text{proj}_v\, \boldsymbol{u} = <4, -3>$
Hence, $|\boldsymbol{u}|\cos 60 = |<4, -3>|$ ✓

$|\boldsymbol{u}| = 10$ ✓

(d) Find $\boldsymbol{u}$.

$\boldsymbol{u} = <4, -3> + \lambda <3, 4>$.

$\Rightarrow |<4 + 3\lambda, -3 + 4\lambda>| = 10$ ✓

$(4 + 3\lambda)^2 + (-3 + 4\lambda)^2 = 100$ ✓

$16 + 24\lambda + 9\lambda^2 + 9 - 24\lambda + 16\lambda^2 = 100$

$25\lambda^2 = 75$

$\lambda = \pm\sqrt{3}$. ✓

Hence, $\boldsymbol{u} = <4, 3> \pm \sqrt{3}<3, 4>$. ✓

Calculator Assumed

6. [11 marks: 1, 3, 2, 3, 2]

Let $\boldsymbol{u} = <2, 1>$, $\boldsymbol{v} = <1, -2>$ and $\boldsymbol{w} = <3, 9>$.

(a) Show that $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular.

> $<2, 1> \bullet <1, -2> = 0$. ✓
> Hence, $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular.

(b) Given that $\boldsymbol{w} = m\boldsymbol{u} + n\boldsymbol{v}$, find m and n.

> x-components: $2m + n = 3$ ✓
> y-components: $m - 2n = 9$ ✓
> $m = 3, n = -3$ ✓
> $\begin{cases} 2x+y=3 \\ x-2y=9 \end{cases} \Big|_{x,y}$ $\{x=3, y=-3\}$

(c) Find the vector projection of $\boldsymbol{w}$ onto $<-2, -1>$.

> From (b), $3\boldsymbol{u}$ and $-3\boldsymbol{v}$ are components of $\boldsymbol{w}$ parallel and perpendicular to $<2, 1>$ respectively. ✓
> Since $<-2, -1>$ is parallel to $\boldsymbol{u}$,
> vector projection of $\boldsymbol{w}$ onto $<-2, -1>$ is $3\boldsymbol{u} = <6, 3>$. ✓

(d) $\boldsymbol{w}$ is the vector projection of $\lambda <2, 1>$ onto $\boldsymbol{w}$. Find λ.

> $\hat{w} = \frac{1}{\sqrt{90}} <3, 9>$.
> $\Rightarrow \lambda <2, 1> \bullet \frac{1}{\sqrt{90}} <3, 9> = \sqrt{90}$ ✓✓
> $15\lambda = 90$
> $\lambda = 6$ ✓

(e) If $\boldsymbol{w}$ is the vector projection of $\lambda <2, 1>$ onto $\boldsymbol{w}$, find the component of $\lambda <2, 1>$ that is perpendicular to $\boldsymbol{w}$.

> Let required vector be $\boldsymbol{a}$.
> Hence, $\boldsymbol{a} + \boldsymbol{w} = \lambda <2, 1> = 6 <2, 1>$ ✓
> $\boldsymbol{a} = <12, 6> - <3, 9>$
> $= <9, -3>$. ✓

Calculator Assumed

7. [7 marks: 5, 2]

The vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is $\frac{1}{2}\boldsymbol{b}$. The vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\boldsymbol{a}$.

(a) Show mathematically that $|\boldsymbol{b}|^2 = 2|\boldsymbol{a}|^2$.

> $|\text{proj}_b\, \boldsymbol{a}| = \boldsymbol{a} \bullet \hat{\boldsymbol{b}} = \boldsymbol{a} \bullet \frac{1}{|\boldsymbol{b}|}\boldsymbol{b}$
> But $|\text{proj}_b\, \boldsymbol{a}| = |\frac{1}{2}\boldsymbol{b}|$
> $\Rightarrow \boldsymbol{a} \bullet \frac{1}{|\boldsymbol{b}|}\boldsymbol{b} = \frac{1}{2}|\boldsymbol{b}|$ ✓
> $\boldsymbol{a} \bullet \boldsymbol{b} = \frac{1}{2}|\boldsymbol{b}|^2$ I ✓
>
> $|\text{proj}_a\, \boldsymbol{b}| = \boldsymbol{b} \bullet \hat{\boldsymbol{a}} = \boldsymbol{b} \bullet \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$
> But $|\text{proj}_a\, \boldsymbol{b}| = |\boldsymbol{a}|$
> $\Rightarrow \boldsymbol{b} \bullet \frac{1}{|\boldsymbol{a}|}\boldsymbol{a} = |\boldsymbol{a}|$ ✓
> $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}|^2$ II ✓
>
> From I & II: $\frac{1}{2}|\boldsymbol{b}|^2 = |\boldsymbol{a}|^2$ ✓
> $|\boldsymbol{b}|^2 = 2|\boldsymbol{a}|^2$

(b) Find the acute angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

> Let the acute angle be θ.
> $\cos\theta = \frac{\boldsymbol{a} \bullet \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$
> $= \frac{|\boldsymbol{a}|^2}{|\boldsymbol{a}| \times \sqrt{2}|\boldsymbol{a}|}$ ✓
> $= \frac{1}{\sqrt{2}}$
> $\theta = 45°$ ✓

14 Scalar Product III

Calculator Assumed

1. [6 marks: 2, 1, 3]

Given that vector ***a*** has magnitude 2 ms^{-1} in the direction 060°, ***b*** has magnitude 4 ms^{-1} in the direction 045° :

(a) Find ***a*** and ***b*** in the form $x\,\mathbf{i} + y\,\mathbf{j}$.

$\mathbf{a} = <2\cos 30°, 2\sin 30°>$
$= <\sqrt{3}, 1>$ ✓
$\mathbf{b} = <4\cos 45°, 4\sin 45°>$
$= <2\sqrt{2}, 2\sqrt{2}>$ ✓

toRect([2,∠(30)]) $[\sqrt{3}\ \ 1]$
toRect([4,∠(45)]) $[2\cdot\sqrt{2}\ \ 2\cdot\sqrt{2}]$

(b) Find ***a*** • ***b***.

$\mathbf{a} \bullet \mathbf{b} = <\sqrt{3}, 1> \bullet <2\sqrt{2}, 2\sqrt{2}>$

$= 2(\sqrt{2} + \sqrt{6})$ ✓

dotP([√3 1],[2·√2 2·√2]) $2\cdot\sqrt{6}+2\cdot\sqrt{2}$

(c) Use your answers in (a) and/or (b) to find cos 15° in exact form.

Angle between ***a*** and ***b*** is 15°. ✓

Hence, $\cos 15° = \dfrac{<\sqrt{3},1>\bullet<2\sqrt{2},2\sqrt{2}>}{|<\sqrt{3},1>||<2\sqrt{2},2\sqrt{2}>|}$ ✓

$= \dfrac{2(\sqrt{2}+\sqrt{6})}{2\times 4}$

$= \dfrac{\sqrt{2}+\sqrt{6}}{4}$ ✓

Calculator Assumed

2. [6 marks: 1, 5]

A 12 noon, A is at the point with position vector 3i + 2j km and moving with velocity 4i – 2j kmh^{-1}. B is a stationary object at the point with position vector 3i – j km.

(a) Find the position vector of A at time *t* hours after 12 noon.

$\mathbf{OA} = \begin{pmatrix}3\\2\end{pmatrix} + t\begin{pmatrix}4\\-2\end{pmatrix}$. ✓

(b) Use a scalar product method to find when A is closest to B. State this distance.

BA = **OA** – **OB**

$= \left[\begin{pmatrix}3\\2\end{pmatrix} + t\begin{pmatrix}4\\-2\end{pmatrix}\right] - \begin{pmatrix}3\\-1\end{pmatrix}$

$= \begin{pmatrix}4t\\3-2t\end{pmatrix}$ ✓

BA is perpendicular to (4i – 2j) .
Hence, **BA** . (4i – 2j) = 0 .

$\Rightarrow \begin{pmatrix}4t\\3-2t\end{pmatrix}\bullet\begin{pmatrix}4\\-2\end{pmatrix} = 0$ ✓

$16t - 6 + 4t = 0$
$t = 0.3$ ✓

Since, $t = 0.3$, **BA** = 1.2i + 2.4j .
Perpendicular distance = |**BA**|
$= \sqrt{(1.2^2 + 2.4^2)}$
$= 2.6833$ ✓

A is closest to B at (12 + 0.3 × 60) = 12.18 pm at a distance of 2.68 km. ✓

Calculator Assumed

3. [8 marks: 1, 1, 2, 4]

At 0600 hours, P is at the point with position vector $-10\mathbf{i} - 20\mathbf{j}$ km and moving with velocity $12\mathbf{i} + 2\mathbf{j}$ kmh^{-1}. At the same instant, Q is at a point with position vector $2\mathbf{i} + 10\mathbf{j}$ km and moving with velocity $10\mathbf{i} - 7\mathbf{j}$ kmh^{-1}.

(a) Find the position vector of P relative to Q at 0600 hours.

$${}_P\mathbf{r}_Q = \mathbf{QP} = \mathbf{OP} - \mathbf{OQ} = \begin{pmatrix}-10\\-20\end{pmatrix} - \begin{pmatrix}2\\10\end{pmatrix} = \begin{pmatrix}-12\\-30\end{pmatrix} \checkmark$$

(b) Find the velocity of P relative to Q.

$${}_P\mathbf{v}_Q = \mathbf{v}_P - \mathbf{v}_Q = \begin{pmatrix}12\\2\end{pmatrix} - \begin{pmatrix}10\\-7\end{pmatrix} = \begin{pmatrix}2\\9\end{pmatrix} \checkmark$$

(c) Find the position vector of P relative to Q t hours after 0600 hours.

$${}_P\mathbf{r}_Q(t) = \begin{pmatrix}-12\\-30\end{pmatrix} + t\begin{pmatrix}2\\9\end{pmatrix} = \begin{pmatrix}-12+2t\\-30+9t\end{pmatrix} \checkmark\checkmark$$

(d) Use your answers in (b) and (c) to find the closest distance between P and Q. State when this occurs.

At closest approach, ${}_P\mathbf{r}_Q(t)$ is perpendicular to ${}_P\mathbf{v}_Q$.

Hence, $\begin{pmatrix}-12+2t\\-30+9t\end{pmatrix} \bullet \begin{pmatrix}2\\9\end{pmatrix} = 0$ ✓

$\Rightarrow\ -24 + 4t - 270 + 81t = 0$

$t = 3.4588$ ✓

When $t = 3.4588$, ${}_P\mathbf{r}_Q(t) = \begin{pmatrix}-5.0824\\1.1292\end{pmatrix}$

$|{}_P\mathbf{r}_Q| = 5.2063$ ✓

Hence, closest distance between P and Q = 5.21 km.
This occurred at 0600 + 3 hours and 28 minutes = 0928 hours. ✓

Calculator Assumed

4. [10 marks: 2, 3, 1, 4]

At 0800 hours, P is at the point with position vector $10i + 15j$ km and moving with constant velocity $\boldsymbol{v}$ kmh^{-1}. At 0900 hours, Q starts moving from a point with position vector $-10i - 50j$ km with constant velocity $3i + 8j$ km h^{-1}.

(a) Find v, if P arrives at $4i + 0j$ at 1100 hours.

$$\begin{pmatrix}10\\15\end{pmatrix} + 3v = \begin{pmatrix}4\\0\end{pmatrix} \checkmark$$

$$v = \begin{pmatrix}-2\\-5\end{pmatrix} \text{ kmh}^{-1}. \checkmark$$

(b) Find the position vector of P relative to Q, t hours after 0900 hours.

Position vector of P at time t hours:

$$r_P(t) = \begin{pmatrix}10-2-2t\\15-5-5t\end{pmatrix} = \begin{pmatrix}8-2t\\10-5t\end{pmatrix} \checkmark$$

Position vector of Q at time t hours:

$$r_Q(t) = \begin{pmatrix}-10+3t\\-50+8t\end{pmatrix} \checkmark$$

Position vector of P relative to Q:

$${}_P r_Q(t) = \begin{pmatrix}8-2t\\10-5t\end{pmatrix} - \begin{pmatrix}-10+3t\\-50+8t\end{pmatrix} = \begin{pmatrix}18-5t\\60-13t\end{pmatrix} \checkmark$$

(c) Determine the velocity of P relative to Q.

$${}_P v_Q = \begin{pmatrix}-2\\-5\end{pmatrix} - \begin{pmatrix}3\\8\end{pmatrix} = \begin{pmatrix}-5\\-13\end{pmatrix} \checkmark$$

(d) Use your answers in (b) and (c) to find the closest distance between P and Q. State when this occurs.

At closest approach:

${}_P r_Q(t) \bullet {}_P v_Q = 0$

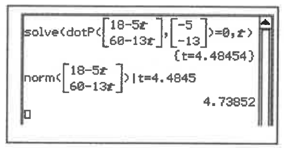
$\begin{pmatrix}18-5t\\60-13t\end{pmatrix} \bullet \begin{pmatrix}-5\\-13\end{pmatrix} = 0$ ✓

$t = 4.4845$ ✓

Closest Distance = $|{}_P r_Q(4.4845)|$

= 4.74 km ✓

Achieved at 1329 hours. ✓



Calculator Assumed

5. [10 marks: 2, 2, 1, 5]

A force F_1 of $< 4, 4 >$ Newtons is applied to a body and causes the body to be displaced by $< 3, 1 >$ metres.

(a) Find the component of the applied force along the direction of motion.

> Component along direction of motion
> $= | < 4, 4 > \bullet \frac{1}{\sqrt{10}} < 3, 1 > | \frac{1}{\sqrt{10}} < 3, 1 >$ ✓
> $= \frac{8}{5} < 3, 1 >$ ✓

(b) Find the component of the applied force perpendicular to the direction of motion.

> Component perpendicular to direction of motion
> $= < 4, 4 > - \frac{8}{5} < 3, 1 >$ ✓
> $= \frac{4}{5} < -1, 3 >$ ✓

(c) Determine the work done by the applied force.

> Work done $= < 4, 4 > \bullet < 3, 1 >$
> $= 16$ Joules. ✓

(d) Another force F_2 of $< x, y >$ Newtons is applied to the same body. But half as much work is required to cause the same displacement to the body.

(i) Find a possible pair of values for x and y.

> Work done: $< x, y > \bullet < 3, 1 > = 8$
> $3x + y = 8$ ✓
> Possible F_2: $< 0, 8 >$ ✓

(ii) Find x and y if $|F_2| = 2\sqrt{2}$ Newtons.

> $3x + y = 8$ I
> $|F_2| = 2\sqrt{2} \Rightarrow x^2 + y^2 = 8$ II ✓
> Solve I & II: $x = 2, y = 2$ ✓
> or $x = \frac{14}{5}, y = -\frac{2}{5}$ ✓

Calculator Assumed

6. [8 marks: 2, 1, 5]

Two forces $< 1, 4 >$ Newtons and $< 3, -2 >$ Newtons act on a single body P and causes the body P to be displaced by $< 8, 4 >$.

(a) Find the vector projection of the resultant force onto the displacement of P.

> Resultant force $= < 1, 4 > + < 3, -2 >$
> $= < 4, 2 >$. ✓
> Since resultant force is parallel to displacement vector,
> projection of resultant force onto displacement vector $= < 4, 2 >$. ✓

(b) Find the work done on P.

> Work done $= < 4, 2 > \bullet < 8, 4 >$
> $= 40$ Joules ✓

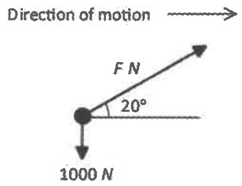
(c) A third force of $< -2, 1 >$ Newtons is applied to P and causes it to move by 10 metres. The work done is 36 Joules. Find the displacement vector.

> Resultant force $= < 4, 2 > + < -2, 1 > = < 2, 3 >$. ✓
> Let displacement vector be $< x, y >$.
> Then, $x^2 + y^2 = 100$ I ✓
> Also $< 2, 3 > \bullet < x, y > = 36$
> $2x + 3y = 36$ II ✓
> Solve I & II $x = 6, y = 8$
> or $x = \frac{66}{13}, y = \frac{112}{13}$
> Hence, displacement vector is $< 6, 8 >$ or $< \frac{66}{13}, \frac{112}{13} >$ metres. ✓✓

Calculator Assumed

7. [6 marks: 3, 2, 1]

An object of weight 1 000 Newtons is being pulled along a horizontal surface by a force of magnitude F Newtons inclined at an angle of 20° to the surface. The motion of the object is opposed by a horizontal force of magnitude 500 N.



(a) Find in terms of F, the magnitude of the pulling force perpendicular to the surface. Hence, find F.

> Magnitude of pulling force perpendicular to surface
> $= F \sin 20^\circ$ N ✓
>
> Since, there is no motion vertical to the surface,
> $F \sin 20^\circ = 1\,000$ ✓
> $F = 2\,923.8044 \approx 2\,923.8$ N ✓

(b) Find the component of the pulling force in the direction of motion. Hence, find the magnitude of the resultant force in the direction of motion.

> Magnitude of pulling force parallel to surface
> $= F \cos 20^\circ = 2\,747.4774$ ✓
>
> Magnitude of resultant force in direction of motion
> $= 2\,747.4774 - 500$
> $= 2\,247.4774 \approx 2\,247.5$ N ✓

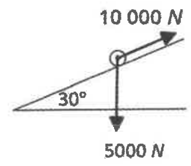
(c) Find the work done by the pulling force in moving the object 50 m in its direction of motion.

> Work done $= 2\,747.4774 \times 50$
> $\approx 137\,373.9$ J or 137.4 kJ ✓

Calculator Assumed

8. [7 marks: 2, 1, 2, 2]

A body of weight 5 000 Newtons is pulled up along a plane inclined at an angle of 30° with the horizontal by a force of magnitude 10 000 N. There is a force of magnitude 500 N opposing the motion of the body.



(a) Find the component of the gravitational force on the body along the inclined plane. Hence, find the magnitude of the resultant force in the direction of the motion of the body.

> Component of gravitational force along inclined plane
> $= 5\,000 \sin 30^\circ = 2\,500$ N ✓
>
> Magnitude of resultant force in direction of motion
> $= 10\,000 - 2\,500 - 500 = 7\,000$ N ✓

(b) Find the work done by the resultant force when the body has moved 10 m along the inclined plane.

> Work done $= 7\,000 \times 10$ J $= 70$ kJ ✓

(c) Find the work done by the resultant force when the body has ascended a vertical distance of 10 m.

> Distance moved along the inclined plane
> $= \dfrac{10}{\sin 30} = 20$ m ✓
> Hence, work done $= 7\,000 \times 20$ J $= 140$ kJ ✓

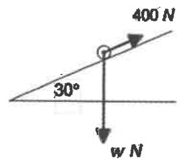
(d) The work done by the resultant force in moving the body up the inclined plane so that the change in its horizontal position is x metres is 280 kJ. Find x.

> Distance moved along plane $= \dfrac{280\,000}{7\,000} = 40$ m. ✓
>
> Hence, $x = 40 \cos 30 = 20\sqrt{3}$ m. ✓

Calculator Assumed

9. [7 marks: 3, 4]

A cart of weight w N is at rest on a set of rail-tracks inclined at an angle of 30° with the horizontal. A force parallel to the inclined plane of magnitude 400 N just prevents the body from slipping down the rail-tracks.



(a) Find the component of the gravitational force acting on the cart along the inclined rail-tracks. Hence, find the weight of the cart.

> Component of gravitational force along plane
> $= w \sin 30^\circ$ ✓
> Since cart is at rest on the plane:
> $w \sin 30^\circ = 400$ ✓
> $w = 800$ N ✓

(b) A force of magnitude 1 000 N at an angle of 30° to the inclined rail-tracks acts on the cart and moves it a distance of 10 m along the tracks. Assume that the cart does not leave the rail-tracks. Find the work done by the resultant force along the inclined plane.

> Component of force along track
> $= 1\,000 \cos 30^\circ = 500\sqrt{3}$ ✓
>
> Magnitude of resultant force along direction of motion
> $= 500\sqrt{3} - 800 \sin 30^\circ$ ✓
> $= 500\sqrt{3} - 400$ N ✓
> Hence, work done $= (500\sqrt{3} - 400) \times 10$ J
> $= 4.66$ kJ ✓

15 Geometric Proofs using Vectors

Calculator Assumed

1. [6 marks: 2, 2, 2,]

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-parallel vectors. Find α and β if:

(a) $2\boldsymbol{a} + (\beta - 2)\boldsymbol{b} = (1 - \alpha)\boldsymbol{a}$

> $1 - \alpha = 2 \Rightarrow \alpha = -1$ ✓
> $\beta - 2 = 0 \Rightarrow \beta = 2$ ✓

(b) $\alpha(3\boldsymbol{a} - 4\boldsymbol{b}) = 6\boldsymbol{a} + \beta\boldsymbol{b}$

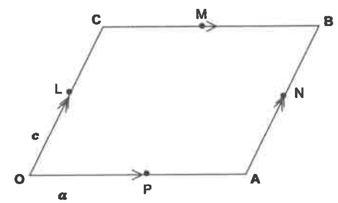
> $3\alpha = 6 \Rightarrow \alpha = 2$ ✓
> $\beta = -4\alpha \Rightarrow \beta = -8$ ✓

(c) $\alpha\boldsymbol{a} + 5\boldsymbol{b}$ is parallel to $3\boldsymbol{a} + \beta\boldsymbol{b}$

> $\alpha\boldsymbol{a} + 5\boldsymbol{b} = \lambda(3\boldsymbol{a} + \beta\boldsymbol{b})$
> $\alpha = 3\lambda \Rightarrow \lambda = \alpha/3$
> $\lambda\beta = 5 \Rightarrow \alpha\beta = 15$ or $\alpha = 15/\beta$ for $\beta \neq 0, \alpha \neq 0$ ✓✓

2. [4 marks: 1, 1, 2]

OABC is a parallelogram. **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$. L, M, N and P are the midpoints of OC, CB, BA and AO respectively.



(a) Find **LM** in terms of **a** and **c**.

> **LM** = **LC** + **CM**
> $= \frac{1}{2}\boldsymbol{c} + \frac{1}{2}\boldsymbol{a}$ ✓

(b) Find **PN** in terms of **a** and **c**.

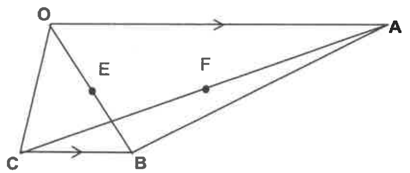
> **PN** = **PA** + **AN**
> $= \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{c}$ ✓

(c) Hence, use a vector method to show that LMNP is a parallelogram.

> Clearly **LM** = **PN**.
> Hence, LMNP has a pair of opposite sides that are parallel and congruent. ✓
> Hence, LMNP is a parallelogram. ✓

Calculator Assumed

3. [8 marks: 2, 4, 2]



OABC is a trapezium with **OA** = 3**CB**. **OA** = ***a*** and **OC** = ***c***. E and F are midpoints of OB and CA respectively.

(a) Find **OE** in terms of ***a*** and ***c***.

$$\mathbf{OE} = \frac{1}{2}\mathbf{OB} = \frac{1}{2}(\mathbf{OC} + \mathbf{CB}) \quad ✓$$
$$= \frac{1}{2}(\boldsymbol{c} + \frac{1}{3}\boldsymbol{a}). \quad ✓$$

(b) Find **EF** in terms of ***a*** and ***c***.

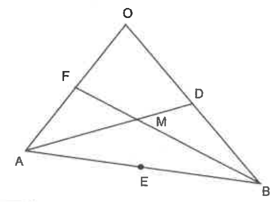
$$\mathbf{AF} = \frac{1}{2}\mathbf{AC} = \frac{1}{2}(\mathbf{AO} + \mathbf{OC}) \quad ✓$$
$$= \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{c}). \quad ✓$$
$$\mathbf{EF} = \mathbf{EO} + \mathbf{OA} + \mathbf{AF}$$
$$= -\frac{1}{2}(\boldsymbol{c} + \frac{1}{3}\boldsymbol{a}) + \boldsymbol{a} + \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{c}) \quad ✓$$
$$= \frac{1}{3}\boldsymbol{a} \quad ✓$$

(c) Prove that CEFB is a parallelogram.

Clearly **EF** = **CB**.
Hence, CEFB has a pair of opposite sides that are parallel and congruent. ✓
Hence, CEFB is a parallelogram. ✓

Calculator Assumed

4. [14 marks: 2, 2, 5, 3, 2]



OAB is a triangle with **OA** = ***a*** and **OB** = ***b***. D, E and F are the midpoints of OB, AB, and OA respectively. **AM** = α**AD** and **MF** = β**BF**.

(a) Find **AD** and **BF** in terms of ***a*** and ***b***.

AD = $-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$ ✓
BF = $\frac{1}{2}\boldsymbol{a} - \boldsymbol{b}$ ✓

(b) Find **AM** and **MF** in terms of ***a*** and ***b***.

$$\mathbf{AM} = \alpha(-\boldsymbol{a} + \tfrac{1}{2}\boldsymbol{b}) = -\alpha\boldsymbol{a} + \frac{\alpha}{2}\boldsymbol{b} \quad ✓$$
$$\mathbf{MF} = \beta(\tfrac{1}{2}\boldsymbol{a} - \boldsymbol{b}) = \frac{\beta}{2}\boldsymbol{a} - \beta\boldsymbol{b} \quad ✓$$

(c) Use your answers in (b) to find α and β.

AF = **AM** + **MF** ✓
$$-\tfrac{1}{2}\boldsymbol{a} = (-\alpha\boldsymbol{a} + \frac{\alpha}{2}\boldsymbol{b}) + (\frac{\beta}{2}\boldsymbol{a} - \beta\boldsymbol{b}) = (-\alpha + \frac{\beta}{2})\boldsymbol{a} + (\frac{\alpha}{2} - \beta)\boldsymbol{b} \quad ✓$$
Compare coefficients for ***b*** and ***a*** vectors:
$$\frac{\alpha}{2} - \beta = 0 \Rightarrow \beta = \frac{\alpha}{2} \quad ✓$$
$$-\alpha + \frac{\beta}{2} = -\frac{1}{2} \Rightarrow -\alpha + \frac{1}{4}\alpha = -\frac{1}{2} \Rightarrow \alpha = \frac{2}{3},\ \beta = \frac{1}{3} \quad ✓✓$$

(d) Show that **OM** = μ**OE** giving the value of μ.

OM = **OA** + **AM**
$$= \boldsymbol{a} + (-\frac{2}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b}) = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b}) \quad ✓$$
OE = **OA** + **AE**
$$= \boldsymbol{a} + \tfrac{1}{2}(-\boldsymbol{a} + \boldsymbol{b}) = \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) \quad ✓$$
Hence, **OM** = $\frac{2}{3}$**OE** ✓

(e) Comment on the significance of the location of M in terms of the lines OE, AD and BF.

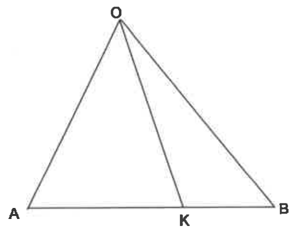
AD and BF are medians of ΔOAB. AD and BF meet at M, such that AM = $\frac{2}{3}$AD and BM = $\frac{2}{3}$BF.

Since OM = $\frac{2}{3}$OE, and OE is also a median, all three medians ✓ meet at M, two-thirds down from the respective vertices. ✓

Calculator Assumed

5. [8 marks: 2, 4, 2]

In triangle OAB, K divides AB in the ratio $\lambda:\mu$ (that is AK:KB = $\lambda:\mu$).



(a) Find **AK** in terms of **OA** and **OB** .

$$\mathbf{AK} = \left[\frac{\lambda}{\lambda+\mu}\right]\mathbf{AB} \quad ✓$$

$$= \left[\frac{\lambda}{\lambda+\mu}\right](-\mathbf{OA}+\mathbf{OB}) \quad ✓$$

(b) Hence, or otherwise. prove that $\mathbf{OK} = \left[\frac{1}{\lambda+\mu}\right][\lambda\mathbf{OB} + \mu\mathbf{OA}]$.

$$\mathbf{OK} = \mathbf{OA} + \mathbf{AK} \quad ✓$$

$$= \mathbf{OA} + \left[\frac{\lambda}{\lambda+\mu}\right](-\mathbf{OA}+\mathbf{OB}) \quad ✓$$

$$= \left[\frac{1}{\lambda+\mu}\right][\,(\lambda+\mu)\,\mathbf{OA} - \lambda\mathbf{OA} + \lambda\mathbf{OB}\,] \quad ✓$$

$$= \left[\frac{1}{\lambda+\mu}\right](\,\mu\mathbf{OA} + \lambda\mathbf{OB}\,) \quad ✓$$

(c) Use the result above to find the position vector of a point that divides the line connecting A (1, 2) to B (6, 12) in the ratio 2: 3 .

$$\text{OK} = \left[\frac{1}{2+3}\right](3\text{OA} + 2\text{OB}) \quad ✓$$

$$= \frac{1}{5}[3<1,2>+2<6,12>]$$

$$= <3,6>. \quad ✓$$

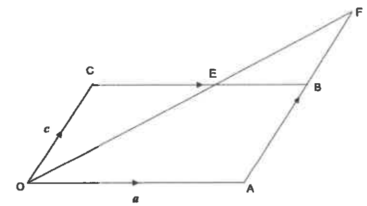
Calculator Assumed

6. [8 marks]

OABC is a parallelogram. The point E divides CB in the ratio $\alpha : \beta$. That is, the point E is such that $\mathbf{EB} = \frac{\beta}{\alpha+\beta}\mathbf{CB}$. OE extended meets the AB extended at F. Use vector methods to prove that:

Area of ΔFEB = $\left(\frac{\beta}{\alpha+\beta}\right)^2$ × Area of ΔFOA.

[Hint: Let **EF** = λ**OF** and **BF** = μ**AF**.]



Let **EF** = λ**OF** and **BF** = μ**AF**.

$$\mathbf{EF} = \mathbf{EB} + \mathbf{BF} \quad ✓$$

$$\lambda\mathbf{OF} = \frac{\beta}{\alpha+\beta}\mathbf{CB} + \mu\mathbf{AF}$$

$$= \frac{\beta}{\alpha+\beta}\mathbf{OA} + \mu(\mathbf{AO}+\mathbf{OF})$$

$$= \frac{\beta}{\alpha+\beta}\mathbf{OA} + \mu(-\mathbf{OA}+\mathbf{OF})$$

$$(\lambda-\mu)\mathbf{OF} = (\frac{\beta}{\alpha+\beta} - \mu)\mathbf{OA} \quad ✓✓$$

Since OF and OA are non-parallel:

$\lambda - \mu = 0 \Rightarrow \lambda = \mu$

and $\frac{\beta}{\alpha+\beta} - \mu = 0 \Rightarrow \mu = \frac{\beta}{\alpha+\beta}$ ✓

Hence, and $\lambda = \frac{\beta}{\alpha+\beta}$. ✓

Hence, $\mathbf{EF} = \frac{\beta}{\alpha+\beta}\mathbf{OF}$ and $\mathbf{BF} = \frac{\beta}{\alpha+\beta}\mathbf{AF}$

Area of ΔFOA = $\frac{1}{2} \times |\mathbf{OF}| \times |\mathbf{AF}| \times \sin E\hat{F}B$ ✓

Area of ΔFEB = $\frac{1}{2} \times |\mathbf{FE}| \times |\mathbf{FB}| \times \sin E\hat{F}B$

$$= \frac{1}{2} \times \frac{\beta}{\alpha+\beta}|\mathbf{OF}| \times \frac{\beta}{\alpha+\beta}|\mathbf{AF}| \times \sin\sin E\hat{F}B \quad ✓$$

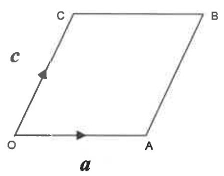
$$= \left(\frac{\beta}{\alpha+\beta}\right)^2 \times \frac{1}{2} \times |\mathbf{OF}| \times |\mathbf{AF}| \times \sin E\hat{F}B \quad ✓$$

$$= \left(\frac{\beta}{\alpha+\beta}\right)^2 \times \text{Area of } \Delta\text{FOA}.$$

Calculator Assumed

7. [4 marks]

OABC is a rhombus. **OA** = $\boldsymbol{a}$ and **OC** = $\boldsymbol{c}$.
Use a vector method to show that the diagonals of a rhombus are perpendicular to each other.



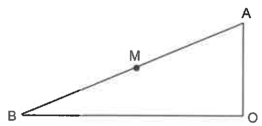
Diagonal **AC** = $-\boldsymbol{a} + \boldsymbol{c}$
Diagonal **OB** = $\boldsymbol{a} + \boldsymbol{c}$

AC • **OB** = $(-\boldsymbol{a} + \boldsymbol{c}) \bullet (\boldsymbol{a} + \boldsymbol{c})$ ✓
$= -\boldsymbol{a} \bullet \boldsymbol{a} - \boldsymbol{a} \bullet \boldsymbol{c} + \boldsymbol{c} \bullet \boldsymbol{a} + \boldsymbol{c} \bullet \boldsymbol{c}$
$= -|\boldsymbol{a}|^2 + |\boldsymbol{c}|^2$ ✓
$= 0$ since for a rhombus $|\boldsymbol{a}| = |\boldsymbol{c}|$. ✓✓

---

8. [8 marks: 1, 3, 4]

OAB is a right angled triangle with ∠ AOB = 90°.
OA = $\boldsymbol{a}$ and **OB** = $\boldsymbol{b}$. M is the midpoint of AB.



(a) Explain why $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

OA is perpendicular to OB.
Hence, **OA . OB** = 0.
Therefore, $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. ✓

(b) Find $|\mathbf{BM}|^2$ in terms of a and b, where $|\boldsymbol{a}| = a$ and $|\boldsymbol{b}| = b$.

BM = $\frac{1}{2}$**BA** = $\frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$ ✓
$|\mathbf{BM}|^2 = \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b}) \cdot \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$ ✓
$= \frac{1}{4}(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\,\boldsymbol{a} \cdot \boldsymbol{b})$
$= \frac{1}{4}(a^2 + b^2)$ since $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ ✓

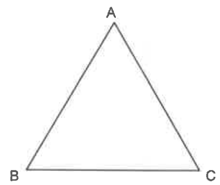
(c) Hence, prove that M is the centre of a circle passing through A, B and O.

OM = $\boldsymbol{b}$ + **BM** = $\boldsymbol{b} + \frac{1}{2}(\boldsymbol{a} - \boldsymbol{b})$
$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$ ✓
$|\mathbf{OM}|^2 = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) \cdot \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$
$= \frac{1}{4}(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\,\boldsymbol{a} \cdot \boldsymbol{b})$
$= \frac{1}{4}(a^2 + b^2)$ since $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ ✓
Hence, $|\mathbf{OM}| = |\mathbf{BM}| = |\mathbf{MA}|$. ✓
Therefore, M is equidistant from O, A and B.
M is then the centre of a circle passing through A, B and O. ✓

Calculator Assumed

9. [12 marks: 2, 3, 4, 3]

ABC is an isosceles triangle with AB = AC.
Also, **BA** = $\boldsymbol{a}$ and **CB** = $\boldsymbol{b}$.



(a) Show that $|\boldsymbol{a} + \boldsymbol{b}| = |\boldsymbol{a}|$

CA = **CB** + **BA**
$= \boldsymbol{b} + \boldsymbol{a}$ ✓
But, $|\mathbf{CA}| = |\mathbf{BA}|$ ✓
$\Rightarrow |\boldsymbol{a} + \boldsymbol{b}| = |\boldsymbol{a}|$

(b) Show that $|\boldsymbol{b}|^2 = -2\,\boldsymbol{a} \cdot \boldsymbol{b}$.

$|\boldsymbol{a} + \boldsymbol{b}| = |\boldsymbol{a}|$
$\Rightarrow (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{a}$ ✓
$|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\,\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|^2$ ✓
$\Rightarrow |\boldsymbol{b}|^2 = -2\,\boldsymbol{a} \cdot \boldsymbol{b}$ ✓

(c) Show that $\cos C = \dfrac{-\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$.

$\cos C = \dfrac{\mathbf{CA} \cdot \mathbf{CB}}{|\mathbf{CA}||\mathbf{CB}|}$
$= \dfrac{(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{b}}{|\boldsymbol{a} + \boldsymbol{b}||\boldsymbol{b}|}$ ✓
$= \dfrac{(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$ since $|\boldsymbol{a} + \boldsymbol{b}| = |\boldsymbol{a}|$ ✓
$= \dfrac{\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$
$= \dfrac{\boldsymbol{a} \cdot \boldsymbol{b} - 2\,\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$ since $|\boldsymbol{b}|^2 = -2\,\boldsymbol{a} \cdot \boldsymbol{b}$ ✓
$= \dfrac{-\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$ ✓

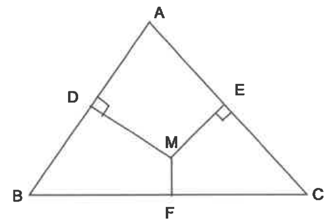
(d) Hence, using a vector method, prove that the base angles of an isosceles triangle are equal.

$\cos B = \dfrac{\mathbf{BA} \cdot \mathbf{BC}}{|\mathbf{BA}||\mathbf{BC}|} = \dfrac{-\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$ ✓
Since cos C = cos B, ∠ACB = ∠ABC. ✓
Hence, the base angles of an isosceles triangle are equal. ✓

Calculator Assumed

10. [10 marks: 1, 2, 2, 3, 2]

DM and EM are respectively the perpendicular bisectors of sides AB and AC of triangle ABC. F is midpoint of BC. Also, **AB** = ***b*** , **AC** = ***c*** and **MD** = ***d***.



(a) Find **ME** in terms of ***b***, ***c*** and ***d***.

> **ME = MD + DA + AE**
> $= \boldsymbol{d} + \frac{1}{2}(-\boldsymbol{b}) + \frac{1}{2}\boldsymbol{c}$
> $= \boldsymbol{d} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b})$ ✓

(b) Use your answer in (a) to show that $[\boldsymbol{d} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b})] \cdot \boldsymbol{c} = 0$

> ME is perpendicular to AC. ✓
> Hence, **ME . AC** = 0 .
> Therefore, $[\boldsymbol{d} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b})] \cdot \boldsymbol{c} = 0$ ✓

(c) Find **MF** in terms of ***b***, ***c*** and ***d***.

> **MF = MD + DB + BF**
> $= \boldsymbol{d} + \frac{1}{2}(\boldsymbol{b}) + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b})$ ✓
> $= \boldsymbol{d} + \frac{1}{2}\boldsymbol{c}$ ✓

(d) Show that **MF . BC** = 0.

> **MF . BC** $= (\boldsymbol{d} + \frac{1}{2}\boldsymbol{c}) \cdot (\boldsymbol{c} - \boldsymbol{b})$ ✓
> $= \boldsymbol{d} \cdot \boldsymbol{c} - \boldsymbol{d} \cdot \boldsymbol{b} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b}) \cdot \boldsymbol{c}$
> $= \boldsymbol{d} \cdot \boldsymbol{c} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b}) \cdot \boldsymbol{c}$ since $\boldsymbol{d} \cdot \boldsymbol{b} = 0$ ✓
> $= [(\boldsymbol{d} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{b})] \cdot \boldsymbol{c}$ ✓
> $= 0$

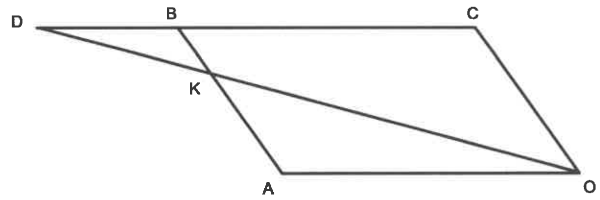
(e) State the significance of the result **MF . BC** = 0 .

> **MF . BC** = 0 ⇒ **MF** is perpendicular to BC.
> But F is the midpoint of BC.
> Hence, **MF** is the perpendicular bisector of BC. ✓
>
> MD, ME are also perpendicular bisectors.
> Hence, the perpendicular bisectors of the sides of a triangle meet at the same point (in this case, M). ✓

Calculator Assumed

11. [5 marks: 2, 3] [TISC]

OABC is a parallelogram with **OA** = ***a*** and **OC** = ***c***.
The point K divides AB in the ratio 2 : 1. OK extended meets the line CB extended at D. **OK** = α**OD** and **CD** = β**CB**.



(a) Find **AK** and **OK** in terms of ***a*** and ***c***.

> $\mathbf{AK} = \frac{2}{3}\mathbf{AB} = \frac{2}{3}\mathbf{OC} = \frac{2}{3}\boldsymbol{c}$. ✓
> **OK = OA + AK**
> $= \boldsymbol{a} + \frac{2}{3}\boldsymbol{c}$ ✓

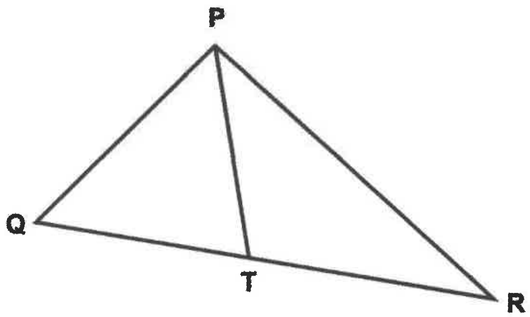
(b) Use vector methods to prove that B divides the line CD in the ratio 2 : 1.

> **OK** = α**OD**
> **OD = OC + CD**
> $= \boldsymbol{c} + \beta\mathbf{CB} = \boldsymbol{c} + \beta(\boldsymbol{a})$
> Hence, **OK** $= \alpha[\boldsymbol{c} + \beta(\boldsymbol{a})]$
> $= \alpha\boldsymbol{c} + \alpha\beta\boldsymbol{a}$. ✓
>
> But from (a), **OK** $= \boldsymbol{a} + \frac{2}{3}\boldsymbol{c}$
>
> Hence, $\alpha\boldsymbol{c} + \alpha\beta\boldsymbol{a} = \boldsymbol{a} + \frac{2}{3}\boldsymbol{c}$ ✓
>
> $\Rightarrow \alpha = \frac{2}{3}$ and $\alpha\beta = 1$.
>
> Therefore, $\beta = \frac{3}{2}$. ✓
> Hence, B divides the line CD in the ratio 2 : 1.

Calculator Assumed

12. [8 marks: 3, 5] [TISC]

In ΔPQR drawn below, the point T is the midpoint of QR. Let **PT** = $\boldsymbol{a}$ and **TR** = $\boldsymbol{b}$.



(a) Find **PR** and **PQ** in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

> **PR** = **PT** + **TR**
> = $\boldsymbol{a} + \boldsymbol{b}$ ✓
> **PQ** = **PT** + **TQ**
> But **TQ** = −**TR** = $-\boldsymbol{b}$ ✓
> Hence, **PQ** = $\boldsymbol{a} - \boldsymbol{b}$ ✓

(b) If T is equidistant to P and R, use a vector method to prove that ∠QPR = 90°.

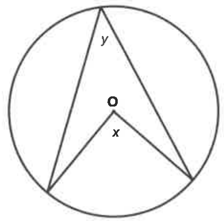
> Since PT = TR ⇒ $|a| = |b|$ ✓
>
> **PR** . **PQ** = $(a + b) . (a - b)$ ✓
> $= a . a - a . b + b . a - b . b$
> $= a . a - b . b$
> $= |a|^2 - |b|^2$ ✓
> $= 0$ ✓
>
> Hence, PR is perpendicular to PQ and ∠QPR = 90°. ✓

16 Geometric Proofs and Circle Properties

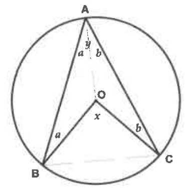
Calculator Assumed

1. [4 marks]

In the accompanying diagram, O is the centre of the circle. Prove that $x = 2y$.

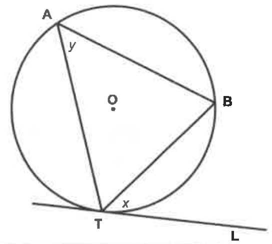


> Let ∠OBA = ∠OAB = a [ΔOAB is isosceles]
> ⇒ ∠BOA = $180^\circ - 2a$. ✓
> Let ∠OCA = ∠OAC = b [ΔOAC is isosceles]
> ⇒ ∠COA = $180^\circ - 2b$. ✓
> Hence, $x = 360^\circ - (180^\circ - 2a) - (180^\circ - 2b)$
> $= 2(a + b)$ ✓
> But ∠BAC = $y = a + b$. ✓
> Hence, $x = 2y$.

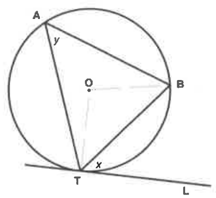


2. [4 marks]

In the accompanying diagram, TL is a tangent to a circle with centre O at T. ∠BTL = x and ∠TAB = y. Prove that $x = y$.



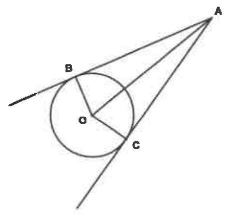
> ∠LTO = 90°. Angle between tangent and radius
> ⇒ ∠OTB = $90^\circ - x$. ✓
> ∠TOB = $180^\circ - 2(90^\circ - x)$
> $= 2x$. ✓
> But 2 × ∠TAB = ∠TOB ✓
> [Angle at centre twice angle at circumference]
> $2 \times y = 2x$. ✓
> Hence, $y = x$.



Calculator Assumed

3. [3 marks]

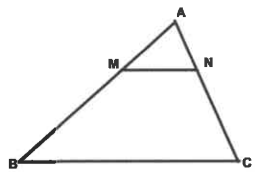
AB and AC are tangents to the circle with centre O.
Prove that ΔABO and ΔACO are congruent.



> ΔABO and ΔACO are right triangles.
> ∠ABO = ∠ACO = 90°. ✓
> BO = CO = radius of circle. ✓
> OA is the common hypotenuse. ✓
> Therefore ΔABO and ΔACO are congruent. (RHS)

4. [7 marks: 3, 2, 2]

In ΔABC, the points M and N divide the sides AB and AC respectively in the ratio 1 : 3.



(a) Prove that ΔAMN and ΔABC are similar.

> $\frac{AM}{AB} = \frac{AN}{AC} = \frac{1}{4}$ (given) ✓✓
> ∠MAN = ∠BAC (common) ✓
> Hence, ΔAMN and ΔABC are similar (SAS).

(b) Hence, deduce that BC = 4MN.

> Since ΔAMN and ΔABC are similar, $\frac{AM}{AB} = \frac{MN}{BC}$. ✓
> But $\frac{AM}{AB} = \frac{1}{4}$. ⇒ $\frac{MN}{BC} = \frac{1}{4}$. ⇒ BC = 4MN. ✓

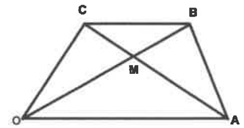
(c) Prove that MN is parallel to BC.

> Since ΔAMN and ΔABC are similar,
> ∠AMN = ∠ABC ✓
> Hence, MN must be parallel to BC.
> These angles being corresponding angles. ✓

Calculator Assumed

5. [5 marks: 3, 2]

OABC is a trapezium with OA parallel to CB.
The diagonals OB and AC intersect at M such that AM : MC = 3 : 1.



(a) Prove that ΔMOA and ΔMBC are similar.

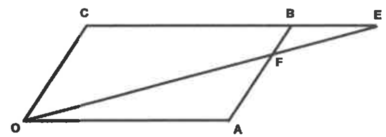
> ∠OMA = ∠BMC (vertically opposite) ✓
> ∠AOM = ∠CBM (alternate angles, OA parallel to CB) ✓
> Hence ΔMOA and ΔMBC are similar (AA). ✓

(b) Hence deduce that OA = 3BC.

> Since ΔMOA and ΔMBC are similar, $\frac{AM}{CM} = \frac{OA}{BC}$. ✓
> But $\frac{AM}{CM} = \frac{3}{1}$. ⇒ $\frac{OA}{BC} = \frac{3}{1}$. ⇒ OA = 3BC. ✓

6. [5 marks: 3, 2]

OABC is a parallelogram with OA parallel and congruent to CB. The point F divides AB in the ratio 2 : 1. OF extended meets the CB extended at E.



(a) Prove that ΔFOA and ΔFEB are similar.

> ∠OFA = ∠EFB (vertically opposite) ✓
> ∠AOF = ∠BEF (alternate angles, OA parallel to BE) ✓
> Hence ΔFOA and ΔFEB are similar (AA). ✓

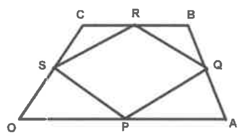
(b) Hence, deduce that F divides OE in the ratio 2 : 1.

> Since ΔFOA and ΔFEB are similar, $\frac{FA}{FB} = \frac{FO}{FE}$. ✓
> But $\frac{FA}{FB} = \frac{2}{1}$. ⇒ $\frac{FO}{FE} = \frac{2}{1}$. ✓
> ⇒ F divides OE in the ratio 2 : 1.

Calculator Assumed

7. [5 marks]

OABC is a trapezium with OA parallel to CB. P, Q, R and S are respectively the midpoints of OA, AB, BC and OC. Prove that the midpoints of the sides of a trapezium form a parallelogram, that is PQRS is a parallelogram.



$\frac{CR}{CB} = \frac{CS}{CO} = \frac{1}{2}$ [Given: R and S midpoints of CB and CO respectively]
$\angle SCR = \angle OCB$ [Common]
Hence, ΔCSR and COB are similar (SAS). ✓
Therefore SR is parallel to OB and $SR = \frac{1}{2}OB$. ✓

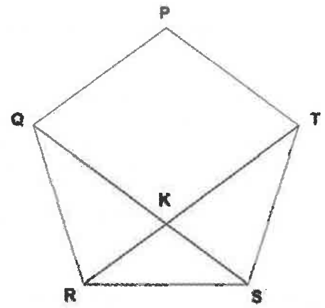
$\frac{AQ}{AB} = \frac{AP}{AO} = \frac{1}{2}$ [Given: Q and P midpoints of AB and AO respectively]
$\angle QAP = \angle BAO$ [Common]
Hence, ΔAPQ and AOB are similar (SAS). ✓
Therefore PQ is parallel to OB and $PQ = \frac{1}{2}OB$. ✓

Hence, SR is congruent and parallel to PQ. $[SR = PQ = \frac{1}{2}OB]$
Therefore, PQRS has a pair of parallel congruent sides, which makes it a parallelogram. ✓

Calculator Assumed

8. [9 marks: 2, 2, 5]

PQRST is a *regular* pentagon of side length 10 cm.



(a) Find the size of $\angle$ RST. Justify your answer.

Since PQRST is a pentagon, the sum of all its interior angles adds up to 540°. ✓
As PQRST is a regular pentagon, all five interior angles are congruent.
Hence, interior angle $\angle RST = \frac{540}{5} = 108°$. ✓

(b) Prove that $\angle STR = 36°$.

SR = ST sides of a regular pentagon
Hence, Δ STR is isosceles. ✓
Hence, $\angle STR = \angle SRT = \frac{180-108}{2} = 36°$. ✓

(c) Find the length of KT. Show clearly how you obtained your answer.

QR = TS Sides of a regular pentagon
RS is common.
$\angle QRS = \angle TSR$ Interior angle of a regular pentagon

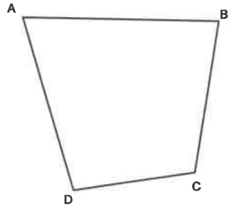
Hence, Δ QRS, TSR are congruent. SAS ✓✓

$\Rightarrow \angle QSR = \angle TRS = 36°$.
$\angle TSK = \angle TSR - \angle QSR = 108 - 36 = 72°$. ✓
In Δ TKS, $\angle TKS = 180 - 36 - 72 = 72°$. Sum angles in a triangle.
Therefore Δ TKS is isosceles. ✓
$\Rightarrow$ TK = TS = 10 cm. ✓

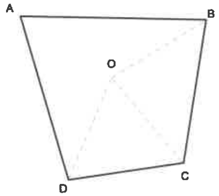
Calculator Assumed

9. [7 marks: 4, 3] [TISC]

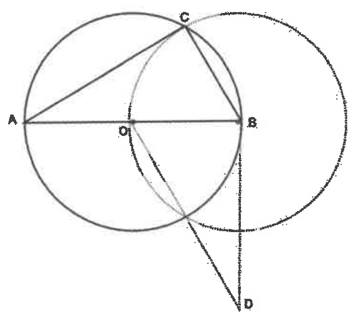
(a) ABCD is a quadrilateral.
$\angle DAB + \angle DCB = 180°$ and
$\angle ADC + \angle ABC = 180°$.
Prove that there is a circle that passes through A, B, C and D.



Let O be the centre of the circle passing through B, C and D.
Let $\angle DCB = x°$.
Then, reflex $\angle DOB = 2x°$. ✓
[Angle at centre = 2 × angle at circumference]
$\Rightarrow$ obtuse $\angle DOB = 360° - 2x° = 2 \times (180° - x°)$. ✓
But given that $\angle DAB = 180° - x°$.
Hence, obtuse $\angle DOB = 2 \times \angle DAB$. ✓
[Angle at centre = 2 × angle at circumference]
Therefore, A could lie on the same circle as B, C and D. ✓



(b) AB is the diameter of the circle with centre O. B is the centre of another circle passing through O. The two circles intersect at C. BD is a tangent to the circle with centre O. If AC = BD, prove that $\angle BOD = \angle CBA$.



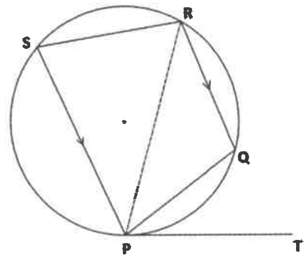
AC = BD	Given	
BC = BO	Radius circle centre B.	✓
$\angle ACB = 90°$	Angle in a semi-circle.	
$\angle OBD = 90°$	BO is a tangent to the circle with radius OB	
$\Rightarrow \angle ACB = \angle OBD$		✓
Hence, ΔBCA and ΔOBD are congruent.	SAS or RHS	✓
Hence, $\angle BOD = \angle CBA$.		

Calculator Assumed

10. [7 marks: 3, 4] [TISC]

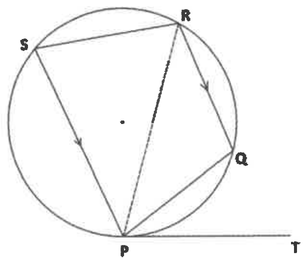
PQRS is a cyclic quadrilateral with RQ parallel to SP. PT is a tangent to the circle. The line PR bisects $\angle SPQ$.

(a) Prove that PQ bisects $\angle TPR$.



Let $\angle SPR = \angle RPQ = \alpha$	Given PR bisects $\angle SPQ$	
But $\angle SPR = \angle PRQ = \alpha$	Alternate angles: RQ parallel to SP.	✓
Also $\angle TPQ = \angle PRQ = \alpha$	Angle between tangent and chord = angle in the opposite segment	✓
Hence $\angle TPQ = \angle QPR = \alpha$		✓
Therefore, PQ bisects $\angle TPR$.		

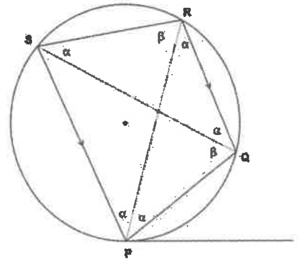
(b) Prove that PQ = SR.
[Hint: Prove that ΔRPQ is congruent to ΔQSR.]



In ΔRPQ and ΔQSR, RQ is common.

Let $\angle SPR = \angle RPQ = \alpha$	Given PR bisects $\angle SPQ$	
$\angle SPR = \angle PRQ = \alpha$	Alternate angles: PQ parallel to SP.	
$\angle SPR = \angle SQR = \alpha$	Angle in the same segment (chord SR)	✓
Hence, $\angle SQR = \angle PRQ = \alpha$		✓
$\angle SRP = \angle SQP = \beta$	Angle in the same segment (chord SP)	✓
Hence $\angle SRQ = \angle PQR = \alpha + \beta$		

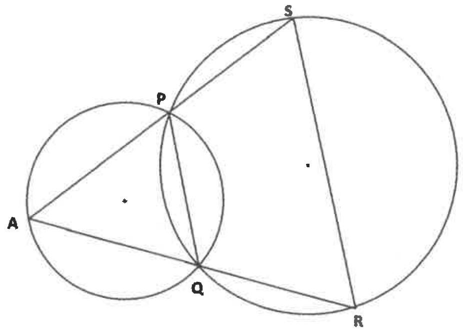
Hence, $\Delta \frac{RPQ}{QSR}$ are congruent (ASA) ✓
$\Rightarrow$ PQ = SR



Calculator Assumed

11. [7 marks: 3, 4]

In the diagram below, the two circles intersect at P and Q. S and P are points on the circumference of the larger circle. The points A, P and S are collinear. The points A, Q and R are collinear.



(a) Prove that ΔAPQ and ΔARS are similar.

Let ∠APQ = $x°$.		
Hence, ∠SPQ = 180 – $x°$.	(Supplementary to ∠PAB)	✓
Then, ∠SRQ = $x°$.	(Opposite angles of a cyclic quadrilateral are supplementary, PQRS is a cyclic quadrilateral)	✓
Hence, ∠APQ = ∠SRQ		
∠PAQ = ∠RAS	(common)	
Hence, ΔAPQ and ΔARS are similar.	(AA)	✓

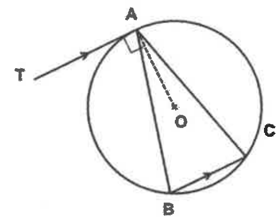
(b) Given that AQ = 10 cm, AP = QR = 8 cm, find PS.

ΔAPQ and ΔARS are similar ⇒	$\frac{AS}{AQ} = \frac{AR}{AP}$	✓
	$\frac{AS}{10} = \frac{18}{8}$	✓
	$AS = 10 \times \frac{18}{8} = \frac{45}{2}$	✓
Therefore	$PS = \frac{45}{2} - 8 = \frac{29}{2}$ cm	✓

Calculator Assumed

12. [3 marks]

The points A, B and C lie on a circle centre O. TA is the tangent to the circle at A. If TA is parallel to BC, prove that ΔABC is isosceles.

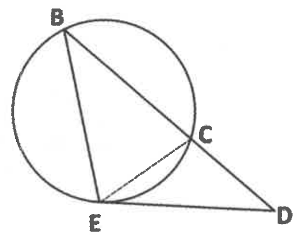


Let ∠BAT = α.		
⇒ ∠ABC = α	Alternate angles: TA parallel to BC	✓
∠ACB = ∠BAT = α	Angle in the alternate segment: TA is a tangent to the circle.	✓
Hence, ∠ACB = ∠ABC = α Therefore ΔABC is isosceles.		✓

13. [5 marks]

The points B, C and E lie on the same circle. The chord BC extended meets the tangent to the circle at E at the point D.

(a) Prove that $ED^2 = CD \times BD$.



Let ∠CDE = α and ∠CED = β.		
∠CBE = ∠CED = β	Alternate segment theorem.	
∠CDE = ∠BDE = α	Common	
Hence Δ CED, EBD are similar (AA).		✓✓
Therefore:	$\frac{CD}{ED} = \frac{ED}{BD}$	✓
	$ED^2 = CD \times BD$	

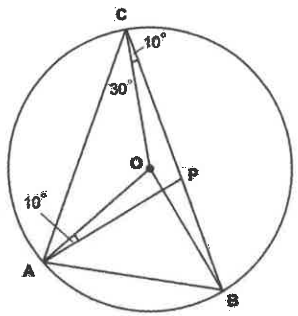
(b) BD is 20 cm long and the point C divides BD in the ratio 3: 2. Hence, find the length of DE.

BC = 12 cm and CD = 8 cm		
Hence	$ED^2 = 8 \times 20$	✓
	$ED = 4\sqrt{10}$ cm	✓

Calculator Assumed

14. [6 marks: 4, 2]

(a) The points A, B and C lie on a circle centre O. $\angle ACO = 30^\circ$ and $\angle BCO = 10^\circ$.
P is a point on the chord BC such that $\angle OAP = 10^\circ$. Find with reasons $\angle APB$.



ΔAOC is isosceles as OA = OC =radius of circle.
$\Rightarrow \angle OAC = \angle OCA = 30^\circ$ Base angles of isosceles ΔBOC ✓✓

$\angle APB$ is external to ΔAPC.
Hence: $\angle APB = \angle PAC + \angle PCA$ ✓
$= 10^\circ + 30^\circ + 30^\circ + 10^\circ$
$= 80^\circ$ ✓

(b) If the points A, B and C lie on the circumference of a circle and O is a point inside the circle, prove or disprove the conjecture that if $\angle AOB = 2\angle ACB$, then O must be the centre of the circle.

Conjecture is False. ✓

Counter-example: See part (a).
$\angle APB = 2\angle ACB$
but P is not the centre of the circle. ✓

17 Trigonometric Equations I (Simple trigonometric ratios)

Calculator Free

1. [0 marks]

Complete the following table. Give answers with rational denominators.

Angle θ in degrees	Angle θ in radians	sin θ	cos θ	tan θ
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	undefined
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	−1
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
180°	π	0	−1	0
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
270°	$\frac{3\pi}{2}$	−1	0	undefined
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	−1
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
360°	2π	0	1	0

Calculator Free

2. [7 marks: 1 each]

Complete the table below. Leave all answers with rational denominators.

Trigonometric Ratio	Value
cosec 60°	$\frac{2\sqrt{3}}{3}$
sec 150°	$-\frac{2\sqrt{3}}{3}$
cot 300°	$-\frac{\sqrt{3}}{3}$
cosec (-135°)	$-\sqrt{2}$
cot $\frac{7\pi}{6}$	$\sqrt{3}$
sec $\frac{4\pi}{3}$	-2
cosec $\left(-\frac{\pi}{4}\right)$	$-\sqrt{2}$

3. [6 marks: 2, 2, 2]

Solve for θ within the given domain:

(a) $\sin\theta = \frac{\sqrt{3}}{2}$ where $0^\circ \le \theta \le 360^\circ$

Hence, $\theta = 60^\circ, 180^\circ - 60^\circ$
$= 60^\circ, 120^\circ$. ✓✓

(b) $\cos\theta = -\frac{\sqrt{2}}{2}$ where $0 \le \theta \le 2\pi$

Hence, $\theta = \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}$
$= \frac{3\pi}{4}, \frac{5\pi}{4}$ ✓✓

(c) $\tan\theta = \sqrt{3}$ where $-\pi < \theta \le \pi$

Hence, $\theta = \frac{\pi}{3}, -\pi + \frac{\pi}{3}$
$= \frac{\pi}{3}, -\frac{2\pi}{3}$ ✓✓

Calculator Free

4. [13 marks: 3, 3, 4, 3]

Given that $\cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$, solve for θ in:

(a) $\cos(\theta + 5^\circ) = \frac{\sqrt{6}+\sqrt{2}}{4}$ for $0 \le \theta \le 360^\circ$

Reference angle for $\theta + 5^\circ = 15^\circ$. ✓
$\theta + 5^\circ$ is in Quadrant 1 and Quadrant 4.
Hence, $\theta + 5^\circ = 15^\circ, 345^\circ$
$\theta = 10^\circ, 340^\circ$ ✓✓

(b) $\cos\theta = -\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)$ for $0 \le \theta \le 360^\circ$

Reference angle for $\theta = 15^\circ$. ✓
θ is in Quadrant 2 and Quadrant 3.
Hence, $\theta = 180^\circ - 15^\circ, 180^\circ + 15^\circ$
$\theta = 165^\circ, 195^\circ$ ✓✓

(c) $\sin\theta = \frac{\sqrt{6}+\sqrt{2}}{4}$ for $0 \le \theta \le 360^\circ$

$\sin\theta = \cos(90^\circ - \theta)$ ✓
Hence: $\cos(90^\circ - \theta) = \frac{\sqrt{6}+\sqrt{2}}{4}$
$90^\circ - \theta$ is in Quadrant 1 and Quadrant 4.
$90^\circ - \theta = 15^\circ, 345^\circ$ ✓
$\theta = 75^\circ, -255^\circ$ ✓
$\theta = 75^\circ, 105^\circ$ ✓

(d) $\sec\theta = \sqrt{6} - \sqrt{2}$ for $0 \le \theta \le 360^\circ$

$\frac{1}{\cos\theta} = \sqrt{6} - \sqrt{2}$ ✓
$\Rightarrow \cos\theta = \frac{1}{\sqrt{6}-\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$ ✓
$\theta = 15^\circ, 345^\circ$ ✓

Calculator Free

5. [11 marks: 1, 3, 4, 3]

Given that $\sin\left(\frac{\pi}{10}\right) = \frac{\sqrt{5}-1}{4}$, find all solutions (in radians) to:

(a) $\sin x = \frac{\sqrt{5}-1}{4}$

$x = (-1)^n \times \left(\frac{\pi}{10}\right) + n\pi \quad n \in \mathbb{Z}$ ✓

(b) $\sin x = \frac{1-\sqrt{5}}{4}$

$\sin x = -\left(\frac{\sqrt{5}-1}{4}\right)$ ✓

$\sin^{-1} x = -\frac{\pi}{10}$ ✓

$x = (-1)^n \times \left(-\frac{\pi}{10}\right) + n\pi \quad n \in \mathbb{Z}$ ✓

(c) $\cos x = \frac{\sqrt{5}-1}{4}$

$\cos x = \sin\left(\frac{\pi}{2} - x\right)$

Hence: $\sin\left(\frac{\pi}{2} - x\right) = \frac{\sqrt{5}-1}{4}$ ✓

$\sin^{-1}\left(\frac{\pi}{2} - x\right) = \frac{\pi}{10}$ ✓

$\left(\frac{\pi}{2} - x\right) = (-1)^n \times \left(\frac{\pi}{10}\right) + n\pi$ ✓

$x = \frac{\pi}{2} - [(-1)^n \times \left(\frac{\pi}{10}\right) + n\pi] \quad n \in \mathbb{Z}$ ✓

(d) $\operatorname{cosec} x = 1 + \sqrt{5}$

$\frac{1}{\sin x} = 1 + \sqrt{5}$ ✓

$\Rightarrow \sin x = \frac{1}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{4}$ ✓

$x = (-1)^n \times \left(\frac{\pi}{10}\right) + n\pi \quad n \in \mathbb{Z}$ ✓

Calculator Free

6. [13 marks: 3, 3, 4, 3]

Solve for all values of θ (in degrees):

(a) $\sin\theta = \cos\theta$

$\sin\theta = \cos\theta \Rightarrow \tan\theta = 1$ ✓

$\tan^{-1}\theta = 45^\circ$ ✓

Hence, $\theta = 45^\circ + 180^\circ n \quad n \in \mathbb{Z}$ ✓

(b) $(\cos\theta - 2)(2\cos\theta + 1) = 0$

$\Rightarrow \cos\theta = 2$ or $\cos\theta = -\frac{1}{2}$ ✓

$\cos\theta = 2$ gives no solution.

Hence, $\cos\theta = -\frac{1}{2}$

$\cos^{-1}\theta = 120^\circ$ ✓

$\Rightarrow \theta = 360^\circ n \pm 120^\circ \quad n \in \mathbb{Z}$ ✓

(c) $2\sin^2\theta + 3\sin\theta - 2 = 0$

Factorise, $(2\sin\theta - 1)(\sin\theta + 2) = 0$ ✓

$\sin\theta = \frac{1}{2}$ or $\sin\theta = -2$ ✓

$\sin\theta = -2$ gives no solution.

Hence, $\sin\theta = \frac{1}{2}$

$\sin^{-1}\theta = 30^\circ$ ✓

$\Rightarrow \theta = (-1)^n \times 30^\circ + 180^\circ n \quad n \in \mathbb{Z}$ ✓

(d) $\sec^2\theta - 4\sec\theta + 4 = 0$

$(\sec\theta - 2)^2 = 0$ ✓

$\sec\theta = 2$

$\cos\theta = \frac{1}{2}$ ✓

$\theta = 360^\circ n \pm 60^\circ \quad n \in \mathbb{Z}$ ✓

18 Trigonometric Identities I (Pythagorean)

Calculator Free

1. [19 marks: 3, 3, 3, 3, 4, 3]

Simplify each of the following expressions:

(a) $\dfrac{1-\cos^2 A}{\tan^2 A}$

$$\text{Expression} \equiv \frac{1-\cos^2 A}{\frac{\sin^2 A}{\cos^2 A}} \quad \checkmark$$
$$\equiv \frac{\sin^2 A}{\frac{\sin^2 A}{\cos^2 A}} \equiv \cos^2 A \quad \checkmark\checkmark$$

(b) $\dfrac{1}{\sin x \tan x + \cos x}$

$$\text{Expression} \equiv \frac{1}{\sin x\left[\frac{\sin x}{\cos x}\right]+\cos x} \quad \checkmark$$
$$\equiv \frac{1}{\frac{\sin^2 x}{\cos x}+\cos x} = \frac{1}{\frac{\sin^2 x+\cos^2 x}{\cos x}} \quad \checkmark$$
$$\equiv \cos x \quad \checkmark$$

(c) $\dfrac{\sin B}{1-\cos B} - \dfrac{1}{\tan B}$

$$\text{Expression} \equiv \frac{\sin B}{1-\cos B} - \frac{\cos B}{\sin B} \quad \checkmark$$
$$\equiv \frac{\sin^2 B-(1-\cos B)\cos B}{(1-\cos B)\sin B} \quad \checkmark$$
$$\equiv \frac{\sin^2 B+\cos^2 B-\cos B}{(1-\cos B)\sin B} \equiv \frac{1}{\sin B} \quad \checkmark$$

(d) $\dfrac{\cos Q}{1+\sin Q} + \dfrac{\cos Q}{1-\sin Q}$

$$\text{Expression} \equiv \frac{\cos Q(1-\sin Q)+\cos Q(1+\sin Q)}{(1+\sin Q)(1-\sin Q)} \quad \checkmark$$
$$\equiv \frac{2\cos Q}{1-\sin^2 Q} \equiv \frac{2\cos Q}{\cos^2 Q} \quad \checkmark$$
$$\equiv \frac{2}{\cos Q} \quad \checkmark$$

Calculator Free

1. (e) $\dfrac{1+\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta}$

$$\text{Expression} \equiv \frac{(1+\sin\theta)^2+\cos^2\theta}{\cos\theta(1+\sin\theta)} \quad \checkmark$$
$$\equiv \frac{1+2\sin\theta+\sin^2\theta+\cos^2\theta}{\cos\theta(1+\sin\theta)} \quad \checkmark$$
$$\equiv \frac{2(1+\sin\theta)}{\cos\theta(1+\sin\theta)} \quad \checkmark$$
$$\equiv \frac{2}{\cos\theta} \quad \checkmark$$

(f) $\dfrac{\sin^3 x-\cos^3 x}{1+\sin x\cos x}$ [Hint: $(a^3-b^3)=(a-b)(a^2+ab+b^2)$]

$$\text{Expression} \equiv \frac{(\sin x-\cos x)(\sin^2 x+\sin x\cos x+\cos^2 x)}{1+\sin x\cos x} \quad \checkmark$$
$$\equiv \frac{(\sin x-\cos x)(1+\sin x\cos x)}{1+\sin x\cos x} \quad \checkmark$$
$$\equiv \sin x-\cos x \quad \checkmark$$

2. [4 marks: 2, 2]

Prove each of the following identities:

(a) $(\sin x+\cos x)^2 = 1+2\sin x\cos x$

$$\text{LHS} \equiv (\sin x+\cos x)^2$$
$$\equiv \sin^2 x+2\sin x\cos x+\cos^2 x \quad \checkmark$$
$$\equiv 1+2\sin x\cos x \quad \checkmark$$
$$\equiv \text{RHS}$$

(b) $\dfrac{1-\sin^2 B}{1-\cos^2 B} = \cot^2 B$

$$\text{LHS} \equiv \frac{1-\sin^2 B}{1-\cos^2 B}$$
$$\equiv \frac{\cos^2 B}{\sin^2 B} = \frac{1}{\tan^2 B} \equiv \cot^2 B \quad \checkmark\checkmark$$
$$\equiv \text{RHS}$$

Calculator Free

3. [11 marks: 3, 4, 4]

(a) Prove $\dfrac{1}{1+\sin\theta}+\dfrac{1}{1-\sin\theta}=2\sec^2\theta$.

$$\begin{aligned}\text{LHS} &\equiv \frac{(1-\sin\theta)+(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} && ✓\\ &\equiv \frac{2}{1-\sin^2\theta} && ✓\\ &\equiv \frac{2}{\cos^2\theta}\equiv 2\sec^2\theta && ✓\\ &\equiv \text{RHS}\end{aligned}$$

(b) Prove $(1+\tan^2 P)(1-\sin^2 P)=1$

$$\begin{aligned}\text{LHS} &\equiv \left[1+\frac{\sin^2 P}{\cos^2 P}\right](1-\sin^2 P) && ✓\\ &\equiv \left[\frac{\cos^2 P+\sin^2 P}{\cos^2 P}\right](\cos^2 P) && ✓✓\\ &\equiv 1 && ✓\\ &\equiv \text{RHS}\end{aligned}$$

(c) Prove $\dfrac{1-2\cos^2 x}{\sin x+\cos x}=\sin x-\cos x$

$$\begin{aligned}\text{LHS} &\equiv \frac{(\cos^2 x+\sin^2 x)-2\cos^2 x}{\sin x+\cos x} && ✓\\ &\equiv \frac{\sin^2 x-\cos^2 x}{\sin x+\cos x} && ✓\\ &\equiv \frac{(\sin x+\cos x)(\sin x-\cos x)}{\sin x+\cos x} && ✓\\ &\equiv \sin x-\cos x && ✓\\ &\equiv \text{RHS}\end{aligned}$$

Calculator Free

4. [10 marks: 4, 3, 3]

(a) Prove $\dfrac{1+\sin M}{\cos M}=\dfrac{\cos M}{1-\sin M}$

$$\begin{aligned}\text{LHS} &\equiv \left[\frac{1+\sin M}{\cos M}\right]\times\left[\frac{\cos M}{\cos M}\right] && ✓\\ &\equiv \left[\frac{(1+\sin M)\cos M}{\cos^2 M}\right]\\ &\equiv \left[\frac{(1+\sin M)\cos M}{1-\sin^2 M}\right] && ✓\\ &\equiv \left[\frac{(1+\sin M)\cos M}{(1+\sin M)(1-\sin M)}\right] && ✓\\ &\equiv \left[\frac{\cos M}{(1-\sin M)}\right]=\text{RHS} && ✓\end{aligned}$$

(b) Prove $\sec x\ \text{cosec}\ x=\tan x+\cot x$

$$\begin{aligned}\text{RHS} &\equiv \frac{\sin x}{\cos x}+\frac{\cos x}{\sin x} && ✓\\ &\equiv \frac{\sin^2 x+\cos^2 x}{\cos x\ \sin x} && ✓\\ &\equiv \frac{1}{\cos x\ \sin x}\equiv \sec x\ \text{cosec}\ x && ✓\\ &\equiv \text{LHS}\end{aligned}$$

(c) Prove $\cos^4 x-\sin^4 x=1-2\sin^2 x$

$$\begin{aligned}\text{LHS} &\equiv (\cos^2 x-\sin^2 x)(\cos^2 x+\sin^2 x) && ✓\\ &\equiv [(1-\sin^2 x)-\sin^2 x]\times 1 && ✓\\ &\equiv 1-2\sin^2 x && ✓\\ &\equiv \text{RHS}\end{aligned}$$

Calculator Free

5. [8 marks: 2, 2, 4]

(a) Prove $\dfrac{1}{1+\cot x} = \dfrac{\tan x}{1+\tan x}$

$$\text{LHS} \equiv \frac{1}{1+\frac{1}{\tan x}} \quad ✓$$
$$\equiv \frac{1}{\left[\frac{\tan x+1}{\tan x}\right]} \quad ✓$$
$$\equiv \frac{\tan x}{1+\tan x} \equiv \text{RHS}$$

(b) Prove $\dfrac{\sin x}{1+\cos x} = \dfrac{1}{\operatorname{cosec} x+\cot x}$

$$\text{RHS} \equiv \frac{1}{\frac{1}{\sin x}+\frac{\cos x}{\sin x}} \quad ✓$$
$$\equiv \frac{1}{\left[\frac{1+\cos x}{\sin x}\right]} \quad ✓$$
$$\equiv \frac{\sin x}{1+\cos x} \equiv \text{LHS}$$

(c) Prove $\dfrac{\cos x}{1+\sin x} = \sec x - \tan x$

$$\text{RHS} \equiv \frac{1}{\cos x} - \frac{\sin x}{\cos x} \quad ✓$$
$$\equiv \frac{1-\sin x}{\cos x} \quad ✓$$
$$\equiv \frac{\cos x(1-\sin x)}{\cos^2 x} \quad ✓$$
$$\equiv \frac{\cos x(1-\sin x)}{1-\sin^2 x}$$
$$\equiv \frac{\cos x(1-\sin x)}{(1-\sin x)(1+\sin x)} \quad ✓$$
$$\equiv \frac{\cos x}{(1+\sin x)}$$
$$\equiv \text{LHS}$$

Calculator Free

6. [7 marks: 4, 3]

Prove each of the following:

(a) $\dfrac{\operatorname{cosec} x+1}{\operatorname{cosec} x-1} = \tan^2 x + 2\tan x\sec x + \sec^2 x$

$$\text{LHS} \equiv \frac{\operatorname{cosec} x+1}{\operatorname{cosec} x-1}$$
$$\equiv \frac{\left[\frac{1}{\sin x}+1\right]}{\left[\frac{1}{\sin x}-1\right]} \quad ✓$$
$$\equiv \frac{1+\sin x}{1-\sin x}$$
$$\equiv \frac{1+\sin x}{1-\sin x} \times \frac{1+\sin x}{1+\sin x} \quad ✓$$
$$\equiv \frac{\sin^2 x + 2\sin x + 1}{1-\sin^2 x} \quad ✓$$
$$\equiv \frac{\sin^2 x + 2\sin x + 1}{\cos^2 x} \quad ✓$$
$$\equiv \tan^2 x + 2\tan x\sec x + \sec^2 x$$
$$\equiv \text{RHS}$$

(b) $\dfrac{1}{\sec^2 x-1} = \operatorname{cosec}^2 x - 1$

$$\text{LHS} \equiv \frac{1}{\sec^2 x-1}$$
$$\equiv \frac{1}{1+\tan^2 x-1} \quad ✓$$
$$\equiv \frac{1}{\tan^2 x}$$
$$\equiv \cot^2 x \quad ✓$$
$$\equiv \operatorname{cosec}^2 x - 1 \quad ✓$$
$$\equiv \text{RHS}$$

19 Trigonometric Identities II (Add/Sub Formulae)

Calculator Free

1. [10 marks: 3, 3, 4]

Use an appropriate trigonometric identity to find the exact value of :

(a) $\sin 75^\circ$

$$\begin{aligned}\sin 75^\circ &= \sin(30^\circ + 45^\circ)\\ &= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ \quad ✓\\ &= \frac{1}{2}\times\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} \quad ✓\\ &= \frac{\sqrt{2}(1+\sqrt{3})}{4} \quad ✓\end{aligned}$$

(b) $\cos 165^\circ$

$$\begin{aligned}\cos 165^\circ &= \cos(120^\circ + 45^\circ)\\ &= \cos 120^\circ \cos 45^\circ - \sin 120^\circ \sin 45^\circ \quad ✓\\ &= -\frac{1}{2}\times\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} \quad ✓\\ &= -\frac{\sqrt{2}(1+\sqrt{3})}{4} \quad ✓\end{aligned}$$

(c) $\tan \frac{7\pi}{12}$

$$\begin{aligned}\tan\frac{7\pi}{12} &= \tan\left[\frac{\pi}{3}+\frac{\pi}{4}\right]\\ &= \frac{\tan\frac{\pi}{3}+\tan\frac{\pi}{4}}{1-\tan\frac{\pi}{3}\tan\frac{\pi}{4}} \quad ✓\\ &= \frac{\sqrt{3}+1}{1-\sqrt{3}} = \frac{\sqrt{3}+1}{1-\sqrt{3}}\times\frac{1+\sqrt{3}}{1+\sqrt{3}} \quad ✓\\ &= \frac{(\sqrt{3}+1)^2}{-2} \quad ✓\\ &= \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3} \quad ✓\end{aligned}$$

Calculator Free

2. [8 marks: 1, 1, 3, 3]

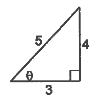
Given that $\sin A = \frac{4}{5}$ and $0 < A < \frac{\pi}{2}$, find the exact value of:

(a) $\cos A$

A is an acute angle.

From the triangle sketched, with $\sin A = \frac{4}{5}$,

$\cos A = \frac{3}{5}$ ✓



(b) $\tan A$

$\tan A = \frac{4}{3}$ ✓

(c) $\sin\left(\frac{\pi}{2} + A\right)$

$$\begin{aligned}\sin\left(\frac{\pi}{2}+A\right) &= \sin\frac{\pi}{2}\cos A + \cos\frac{\pi}{2}\sin A \quad ✓\\ &= \cos A \quad ✓\\ &= \frac{3}{5} \quad ✓\end{aligned}$$

(d) $\cos\left(\frac{\pi}{4} - A\right)$

$$\begin{aligned}\cos\left(\frac{\pi}{4}-A\right) &= \cos\frac{\pi}{4}\cos A + \sin\frac{\pi}{4}\sin A \quad ✓\\ &= \frac{\sqrt{2}}{2}\times\frac{3}{5} + \frac{\sqrt{2}}{2}\times\frac{4}{5} \quad ✓\\ &= \frac{\sqrt{2}}{2}\left(\frac{3}{5}+\frac{4}{5}\right)\\ &= \frac{7\sqrt{2}}{10} \quad ✓\end{aligned}$$

Calculator Assumed

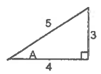
3. [10 marks: 1, 1, 2, 2, 4]

Given that $\sin A = \frac{3}{5}$ and $\cos B = \frac{1}{4}$, where A and B are acute, use appropriate trigonometric identities (relationships) to find the exact value of:

(a) cos A

From the triangle sketched, with $\sin A = \frac{3}{5}$,

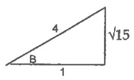
$\Rightarrow \cos A = \frac{4}{5}$ ✓



(b) sin B

From the triangle sketched, with $\cos B = \frac{1}{4}$,

$\Rightarrow \sin B = \frac{\sqrt{15}}{4}$ ✓



(c) sin(A + B)

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$= \frac{3}{5} \times \frac{1}{4} + \frac{4}{5} \times \frac{\sqrt{15}}{4}$ ✓

$= \frac{3 + 4\sqrt{15}}{20}$ ✓

(d) cos (A – B)

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$= \frac{4}{5} \times \frac{1}{4} + \frac{3}{5} \times \frac{\sqrt{15}}{4}$ ✓

$= \frac{4 + 3\sqrt{15}}{20}$ ✓

(e) tan (A + B)

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{\frac{3}{4} + \sqrt{15}}{1 - \frac{3}{4}\sqrt{15}} = \frac{3 + 4\sqrt{15}}{4 - 3\sqrt{15}}$ ✓✓

$= \frac{3 + 4\sqrt{15}}{4 - 3\sqrt{15}} \times \frac{4 + 3\sqrt{15}}{4 + 3\sqrt{15}}$

$= \frac{192 + 25\sqrt{15}}{-119}$ ✓✓

Calculator Assumed

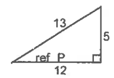
4. [9 marks: 1, 1, 2, 2, 3]

Given that $\sin P = \frac{5}{13}$ and $\cos Q = -\frac{15}{17}$, where $\frac{\pi}{2} \le P \le \pi$ and $\frac{\pi}{2} \le Q \le \pi$, use appropriate trigonometric identities to find the exact value of:

(a) cos P

P is an obtuse angle. From the triangle sketched,

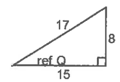
with sin (reference angle for P) $= \frac{5}{13}$, $\cos P = -\frac{12}{13}$ ✓



(b) sin Q

Q is an obtuse angle. From the triangle sketched,

with cos (reference angle for Q) $= \frac{15}{17}$, $\sin Q = \frac{8}{17}$ ✓



(c) sin(P – Q)

$\sin(P - Q) = \sin P \cos Q - \cos P \sin Q$

$= \frac{5}{13} \times (-\frac{15}{17}) - (-\frac{12}{13}) \times \frac{8}{17}$ ✓

$= \frac{21}{221}$ ✓

(d) cos (P + Q)

$\cos(P + Q) = \cos P \cos Q - \sin P \sin Q$

$= (-\frac{12}{13}) \times (-\frac{15}{17}) - \frac{5}{13} \times \frac{8}{17}$ ✓

$= \frac{140}{221}$ ✓

(e) tan (P – Q)

$\tan(P - Q) = \frac{\tan P - \tan Q}{1 + \tan P \tan Q}$

$= \frac{\left[-\frac{5}{12}\right] - \left[-\frac{8}{15}\right]}{1 + \left[-\frac{5}{12}\right] \times \left[-\frac{8}{15}\right]}$ ✓✓

$= \frac{\frac{7}{60}}{\frac{11}{9}} = \frac{21}{220}$ ✓

Calculator Assumed

5. [10 marks: 2, 3, 5]

(a) Prove that sin (–A) = – sin A

$$\sin(0 - A) = \sin 0 \cos A - \cos 0 \sin A \quad ✓$$
$$= -\sin A \quad ✓$$

(b) Prove that $\dfrac{\sin(A-B)}{\sin A \sin B} = \cot B - \cot A$

$$\text{LHS} \equiv \frac{\sin A \cos B + \cos A \sin B}{\sin A \sin B} \quad ✓$$
$$\equiv \frac{\cos B}{\sin B} - \frac{\cos A}{\sin A} \quad ✓$$
$$\equiv \frac{1}{\tan B} - \frac{1}{\tan A} \quad ✓$$
$$\equiv \cot B - \cot A \equiv \text{RHS}$$

(c) Use your answers in parts (a) and (b) to rewrite $\dfrac{\sin 2x}{\sin x \sin 3x} + \dfrac{\sin 2x}{\sin 3x \sin 5x}$

in the form $\dfrac{\sin a}{\sin b \sin c}$, giving the values of a, b and c.

$$\frac{\sin 2x}{\sin x \sin 3x} + \frac{\sin 2x}{\sin 3x \sin 5x} = \frac{-\sin(x-3x)}{\sin x \sin 3x} + \frac{-\sin(3x-5x)}{\sin 3x \sin 5x} \quad ✓✓$$
$$= -(\cot 3x - \cot x) - (\cot 5x - \cot 3x) \quad ✓✓$$
$$= \cot x - \cot 5x$$
$$= \frac{\sin 4x}{\sin x \sin 5x} \quad ✓$$

20 Trigonometric Identities III (Double angle)

Calculator Free

1. [11 marks: 1, 1, 3, 3, 3]

Given that $\sin P = \dfrac{1}{4}$ and $\cos Q = \dfrac{2}{3}$, where $\dfrac{\pi}{2} \le P \le \pi$ and $\dfrac{3\pi}{2} \le Q \le 2\pi$, find the exact value of:

(a) cos P

$$\cos P = -\frac{\sqrt{15}}{4} \quad ✓$$

(b) sin Q

$$\sin Q = -\frac{\sqrt{5}}{3} \quad ✓$$

(c) cos (P + Q)

$$\cos(P+Q) = \cos P \cos Q - \sin P \sin Q \quad ✓$$
$$= -\frac{\sqrt{15}}{4} \times \frac{2}{3} - \frac{1}{4} \times -\frac{\sqrt{5}}{3} \quad ✓$$
$$= \frac{\sqrt{5}(1-2\sqrt{3})}{12} \quad ✓$$

(d) tan 2Q

$$\tan 2Q = \frac{2\tan Q}{1-\tan^2 Q} \quad ✓$$
$$= \frac{2\left(-\frac{\sqrt{5}}{2}\right)}{1-\left(-\frac{\sqrt{5}}{2}\right)^2} \quad ✓$$
$$= 4\sqrt{5} \quad ✓$$

(e) $\sin \dfrac{Q}{2}$

$$\sin^2 \frac{Q}{2} = \tfrac{1}{2}(1 - \cos Q)$$
$$= \tfrac{1}{2}(1 - \tfrac{2}{3}) \quad ✓$$
$$\sin \frac{Q}{2} = \pm\frac{\sqrt{6}}{6} \quad ✓$$
$$\text{But } \frac{3\pi}{4} \le \frac{Q}{2} \le \pi, \Rightarrow \sin\frac{Q}{2} = \frac{\sqrt{6}}{6} \quad ✓$$

Calculator Free

2. [15 marks: 3, 3, 4, 5]

Prove each of the following identities:

(a) $\cos^4 x - \sin^4 x = \cos 2x$

$$\begin{aligned} \text{LHS} &\equiv (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x) \quad ✓ \\ &\equiv (\cos 2x) \times 1 \quad ✓✓ \\ &\equiv \cos 2x \equiv \text{RHS} \end{aligned}$$

(b) $\dfrac{\sin 2t}{1+\cos 2t} = \tan t$

$$\begin{aligned} \text{LHS} &\equiv \frac{2\sin t\cos t}{1+(2\cos^2 t - 1)} \quad ✓ \\ &\equiv \frac{2\sin t\cos t}{2\cos^2 t} \quad ✓ \\ &\equiv \frac{\sin t}{\cos t} \quad ✓ \\ &\equiv \tan t \\ &\equiv \text{RHS} \end{aligned}$$

(c) $\cos 6x = 4\cos^3 2x - 3\cos 2x$

$$\begin{aligned} \text{LHS} &\equiv \cos(2x+4x) \\ &\equiv \cos 2x\cos 4x - \sin 2x \sin 4x \quad ✓ \\ &\equiv \cos 2x\,(2\cos^2 2x - 1) - \sin 2x\,(2\sin 2x\cos 2x) \quad ✓✓ \\ &\equiv 2\cos^3 2x - \cos 2x - 2\sin^2 2x\cos 2x \\ &\equiv 2\cos^3 2x - \cos 2x - 2(1-\cos^2 2x)\cos 2x \quad ✓ \\ &\equiv 4\cos^3 2x - 3\cos 2x \\ &\equiv \text{RHS} \end{aligned}$$

(d) $\dfrac{1-\sin 2t}{\cos 2t} = \dfrac{1-\tan t}{1+\tan t}$

$$\begin{aligned} \text{RHS} &\equiv \frac{1-\frac{\sin t}{\cos t}}{1+\frac{\sin t}{\cos t}} \equiv \frac{\cos t - \sin t}{\cos t + \sin t} \quad ✓✓ \\ &\equiv \frac{\cos t - \sin t}{\cos t + \sin t} \times \frac{\cos t - \sin t}{\cos t - \sin t} \equiv \frac{\cos^2 t + \sin^2 t - 2\sin t\cos t}{\cos^2 t - \sin^2 t} \quad ✓✓✓ \\ &\equiv \frac{1-\sin 2t}{\cos 2t} \equiv \text{LHS} \end{aligned}$$

Calculator Free

3. [11 marks: 3, 4, 4]

Prove each of the following:

(a) $\dfrac{\cos x - \sin 2x}{\cos 2x + \sin x - 1} = \cot x$

$$\begin{aligned} \text{LHS} &\equiv \frac{\cos x - 2\sin x\cos x}{(1-2\sin^2 x) + \sin x - 1} \quad ✓ \\ &\equiv \frac{\cos x(1-2\sin x)}{\sin x(1-2\sin x)} \quad ✓✓ \\ &\equiv \frac{1}{\tan x} \\ &\equiv \cot x \equiv \text{RHS} \end{aligned}$$

(b) $\cot\theta + \tan\theta = 2\operatorname{cosec} 2\theta$

$$\begin{aligned} \text{LHS} &\equiv \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \quad ✓ \\ &\equiv \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} \quad ✓ \\ &\equiv \frac{1}{\sin\theta\cos\theta} \quad ✓ \\ &\equiv \frac{2}{\sin 2\theta} \quad ✓ \\ &\equiv 2\operatorname{cosec} 2\theta \equiv \text{RHS} \end{aligned}$$

(c) $\cot 2x = \dfrac{\cot^2 x - 1}{2\cot x}$

$$\begin{aligned} \text{RHS} &\equiv \frac{\frac{\cos^2 x}{\sin^2 x} - 1}{2\times\frac{\cos x}{\sin x}} \quad ✓ \\ &\equiv \frac{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}}{2\times\frac{\cos x}{\sin x}} \quad ✓ \\ &\equiv \frac{\cos^2 x - \sin^2 x}{2\sin x\cos x} \quad ✓ \\ &\equiv \frac{\cos 2x}{\sin 2x} \quad ✓ \\ &\equiv \cot 2x \equiv \text{LHS} \end{aligned}$$

Calculator Free

4. [9 marks: 3, 3, 3]

(a) Prove that $\sqrt{1-\cos x} = \sqrt{2}\sin\frac{x}{2}$.

$$\text{LHS} \equiv \sqrt{1-\cos[2\times\tfrac{x}{2}]} \quad ✓$$
$$\equiv \sqrt{1-(1-2\sin^2\tfrac{x}{2})} \quad ✓$$
$$\equiv \sqrt{2}\sin\tfrac{x}{2} \equiv \text{RHS} \quad ✓$$

(b) Prove that $\dfrac{\sin x}{1+\cos x} = \tan\dfrac{x}{2}$.

$$\text{LHS} \equiv \frac{\sin[2\times\frac{x}{2}]}{1+\cos[2\times\frac{x}{2}]}$$
$$\equiv \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1+[2\cos^2\frac{x}{2}-1]} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} \quad ✓✓$$
$$\equiv \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$$
$$\equiv \tan\tfrac{x}{2} \equiv \text{RHS} \quad ✓$$

(c) Prove that $\tan^2\left(\dfrac{3x}{2}\right) = \dfrac{1-\cos 3x}{1+\cos 3x}$

$$\text{RHS} \equiv \frac{1-\cos 3x}{1+\cos 3x}$$
$$\equiv \frac{1-\left[1-2\sin^2\left(\frac{3x}{2}\right)\right]}{1+\left[2\cos^2\left(\frac{3x}{2}\right)-1\right]} \quad ✓✓$$
$$\equiv \frac{2\sin^2\left(\frac{3x}{2}\right)}{2\cos^2\left(\frac{3x}{2}\right)} \quad ✓$$
$$\equiv \tan^2\left(\frac{3x}{2}\right)$$
$$= \text{RHS}$$

Calculator Free

5. [13 marks: 4, 3, 6]

(a) Prove that $(a\cos x + b\sin x)^2 + (b\cos x - a\sin x)^2 = a^2 + b^2$.

$$\text{LHS} \equiv (a\cos x + b\sin x)^2 + (b\cos x - a\sin x)^2$$
$$\equiv a^2\cos^2 x + 2ab\sin x\cos x + b^2\sin^2 x + b^2\cos^2 x - 2ab\sin x\cos x + a^2\sin^2 x \quad ✓✓$$
$$\equiv a^2(\cos^2 x + \sin^2 x) + b^2(\sin^2 x + \cos^2 x) \quad ✓✓$$
$$\equiv a^2 + b^2 = \text{RHS}$$

(b) Prove that $\cos^4 2x - \sin^4 2x = \cos 4x$

$$\text{LHS} \equiv \cos^4 2x - \sin^4 2x$$
$$\equiv (\cos^2 2x - \sin^2 2x)(\cos^2 2x + \sin^2 2x) \quad ✓$$
$$\equiv \cos^2 2x - (1-\cos^2 2x) \quad ✓$$
$$\equiv 2\cos^2 2x - 1 \quad ✓$$
$$\equiv \cos 4x \equiv \text{RHS}$$

(c) Prove that $\dfrac{1}{1+\tan x} - \dfrac{1}{1-\tan x} = -\tan 2x$.

$$\text{LHS} \equiv \frac{1}{1+\tan x}\;\frac{1}{1-\tan x}$$
$$\equiv \frac{1}{\left(1+\frac{\sin x}{\cos x}\right)} - \frac{1}{\left(1-\frac{\sin x}{\cos x}\right)} \quad ✓$$
$$\equiv \frac{1}{\left(\frac{\cos x+\sin x}{\cos x}\right)} - \frac{1}{\left(\frac{\cos x-\sin x}{\cos x}\right)} \quad ✓$$
$$\equiv \frac{\cos x}{\cos x+\sin x} - \frac{\cos x}{\cos x-\sin x} \quad ✓$$
$$\equiv \frac{\cos x(\cos x-\sin x) - \cos x(\cos x+\sin x)}{(\cos x+\sin x)(\cos x-\sin x)}$$
$$\equiv \frac{\cos^2 x - \cos x\sin x - \cos^2 x - \cos x\sin x}{(\cos^2 x - \sin^2 x)} \quad ✓$$
$$\equiv \frac{-2\sin x\cos x}{\cos^2 x - \sin^2 x} \quad ✓$$
$$\equiv \frac{-\sin 2x}{\cos 2x}$$
$$\equiv -\tan 2x \equiv \text{RHS} \quad ✓$$

Calculator Free

6. [11 marks: 3, 3, 2, 3]

(a) Prove that $\cos 3A = 4\cos^3 A - 3\cos A$.

$$\begin{aligned}
\text{LHS} &\equiv \cos 3A \\
&\equiv \cos(2A + A) \\
&\equiv \cos 2A \cos A - \sin 2A \sin A \quad ✓\\
&\equiv (2\cos^2 A - 1)\cos A - (2\sin A \cos A)\sin A \quad ✓\\
&\equiv 2\cos^3 A - \cos A - 2\cos A \sin^2 A \\
&\equiv 2\cos^3 A - \cos A - 2\cos A(1 - \cos^2 A) \quad ✓\\
&\equiv 4\cos^3 A - 3\cos A \equiv \text{RHS}
\end{aligned}$$

(b) Prove that $\sin 3A = 3\sin A - 4\sin^3 A$

$$\begin{aligned}
\sin 3A &\equiv \sin(2A + A) \\
&\equiv \sin 2A \cos A + \cos 2A \sin A \quad ✓\\
&\equiv 2\sin A \cos^2 A + (1 - 2\sin^2 A)\sin A \quad ✓\\
&\equiv 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A \quad ✓\\
&\equiv 3\sin A - 4\sin^3 A \equiv \text{RHS}
\end{aligned}$$

(c) Given that $\sin\theta = \frac{1}{4}$, where $0 < \theta < \frac{\pi}{2}$, find:

(i) $\sin 3\theta$

$$\begin{aligned}
\sin 3\theta &= 3\sin\theta - 4\sin^3\theta \\
&= 3 \times \frac{1}{4} - 4 \times \left(\frac{1}{4}\right)^3 \quad ✓\\
&= \frac{3}{4} - \frac{1}{16} = \frac{11}{16} \quad ✓
\end{aligned}$$

(ii) $\cos 3\theta$

$$\begin{aligned}
\cos\theta &= \frac{\sqrt{15}}{4} \quad ✓\\
\cos 3\theta &= 4 \times \left(\frac{\sqrt{15}}{4}\right)^3 - 3 \times \frac{\sqrt{15}}{4} \quad ✓\\
&= \frac{3\sqrt{15}}{16} \quad ✓
\end{aligned}$$

21 Trigonometric Identities IV (Product to Sum and Sum to Product)

Calculator Free

1. [8 marks: 3, 3, 2]

(a) Use an appropriate compound angle formula to prove that

$$\sin\left(\frac{A+B}{2}\right) + \sin\left(\frac{A-B}{2}\right) = 2\sin\frac{A}{2}\cos\frac{B}{2}.$$

$$\begin{aligned}
\text{LHS} &\equiv \sin\left(\frac{A+B}{2}\right) + \sin\left(\frac{A-B}{2}\right) \\
&\equiv \sin\frac{A}{2}\cos\frac{B}{2} + \cos\frac{A}{2}\sin\frac{B}{2} \\
&\quad + \sin\frac{A}{2}\cos\frac{B}{2} - \cos\frac{A}{2}\sin\frac{B}{2} \quad ✓✓\\
&\equiv 2\sin\frac{A}{2}\cos\frac{B}{2} \equiv \text{RHS} \quad ✓
\end{aligned}$$

(b) Use the result in (a) to prove that $\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$.

Let $P + Q = A$ I
$P - Q = B$ II

I + II $P = \frac{A+B}{2}$ III ✓

I − II $Q = \frac{A-B}{2}$ IV ✓

From (a): $\sin\left(\frac{A+B}{2}\right) + \sin\left(\frac{A-B}{2}\right) = 2\sin\frac{A}{2}\cos\frac{B}{2}$

Substitute I, II, III & IV: ✓

Hence: $\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

(c) Use the result in (b) to evaluate $\sin 75^\circ + \sin 15^\circ$

$$\begin{aligned}
\sin 75^\circ + \sin 15^\circ &= 2\sin\left(\frac{75^\circ + 15^\circ}{2}\right)\cos\left(\frac{75^\circ - 15^\circ}{2}\right) \quad ✓\\
&= 2\sin 45^\circ \cos 30^\circ \\
&= 2 \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2} \quad ✓
\end{aligned}$$

Calculator Free

2. [9 marks: 3, 4, 2]

(a) Use an appropriate compound angle formula to prove that

$$\cos\left(\frac{A+B}{2}\right) - \cos\left(\frac{A-B}{2}\right) = -2\sin\frac{A}{2}\sin\frac{B}{2}.$$

$$\text{LHS} \equiv \cos\left(\frac{A+B}{2}\right) - \cos\left(\frac{A-B}{2}\right)$$
$$\equiv \cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2}$$
$$- \cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2} \quad ✓✓$$
$$\equiv -2\sin\frac{A}{2}\sin\frac{B}{2} \equiv \text{RHS} \quad ✓$$

(b) Use the result in (a) to prove that $\cos P - \cos Q = 2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{Q-P}{2}\right)$.

Let $P+Q=A$ I
$P-Q=B$ II

I + II $P = \frac{A+B}{2}$ III ✓

I − II $Q = \frac{A-B}{2}$ IV ✓

From (a): $\cos\left(\frac{A+B}{2}\right) - \cos\left(\frac{A-B}{2}\right) = -2\sin\frac{A}{2}\sin\frac{B}{2}$

Substitute I, II, III & IV:

Hence: $\cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$ V ✓

But $-\sin\left(\frac{P-Q}{2}\right) = \sin\left[-\left(\frac{P-Q}{2}\right)\right]$

$= \sin\left(\frac{Q-P}{2}\right)$ ✓

Hence, V becomes: $\cos P - \cos Q = 2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{Q-P}{2}\right)$

(c) Use the result in (b) to evaluate $\cos 255^\circ - \cos 15^\circ$

$$\cos 255^\circ - \cos 15^\circ = 2\sin\left(\frac{255^\circ + 15^\circ}{2}\right)\sin\left(\frac{15^\circ - 255^\circ}{2}\right) \quad ✓$$
$$= 2\sin 135^\circ \sin(-120^\circ)$$
$$= 2 \times \frac{\sqrt{2}}{2} \times \left(-\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{6}}{2} \quad ✓$$

Calculator Free

3. [11 marks: 2, 3, 2, 2, 2]

(a) Prove that $\cos P\cos Q = \frac{1}{2}[\cos(P+Q) + \cos(P-Q)]$.

$$\text{RHS} \equiv \frac{1}{2}[\cos(P+Q) + \cos(P-Q)]$$
$$\equiv \frac{1}{2}[\cos P\cos Q - \sin P\sin Q + \cos P\cos Q + \sin P\sin Q] \quad ✓$$
$$\equiv \frac{1}{2} \times 2\cos P\cos Q$$
$$\equiv \cos P\cos Q \equiv \text{LHS} \quad ✓$$

(b) Use the result in (a) to prove that $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$.

Let $P = 90^\circ - A$ and $Q = B$. ✓
Hence identity in (a) becomes:

$\cos(90^\circ - A)\cos B = \frac{1}{2}[\cos(90^\circ - A + B) + \cos(90^\circ - A - B)]$

$\cos(90^\circ - A)\cos B = \frac{1}{2}[\cos(90^\circ - (A-B)) + \cos(90^\circ - (A+B))]$ I ✓

Since $\cos(90^\circ - A) = \sin A$, I becomes:

$\sin A\cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$ ✓

(c) Use the results in (a) and/or (b) to evaluate $\cos^2 15^\circ$.

$$\cos 15^\circ\cos 15^\circ = \frac{1}{2}[\cos(15^\circ + 15^\circ) + \cos(15^\circ - 15^\circ)] \quad ✓$$
$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} + 1\right) \quad ✓$$

(d) Use the results in (a) and/or (b) to evaluate $\sin 15^\circ\cos 15^\circ$.

$$\sin 15^\circ\cos 15^\circ = \frac{1}{2}[\sin(15^\circ + 15^\circ) + \sin(15^\circ - 15^\circ)] \quad ✓$$
$$= \frac{1}{4} \quad ✓$$

(e) Hence, evaluate $\cot 15^\circ$.

$$\cot 15^\circ = \frac{\cos 15^\circ\cos 15^\circ}{\sin 15^\circ\cos 15^\circ} \quad ✓$$
$$= \frac{\frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right)}{\frac{1}{4}} = 2 + \sqrt{3} \quad ✓$$

Calculator Free

4. [12 marks: 3, 4, 5]

(a) Prove that $\dfrac{\sin 4A + \sin 2A}{\sin 4A - \sin 2A} = \tan 3A \cot A$.

$$\text{LHS} = \frac{\sin 4A + \sin 2A}{\sin 4A - \sin 2A}$$
$$= \frac{2\sin\left(\frac{4A+2A}{2}\right)\cos\left(\frac{4A-2A}{2}\right)}{2\sin\left(\frac{4A-2A}{2}\right)\cos\left(\frac{4A+2A}{2}\right)} \quad ✓✓$$
$$= \frac{2\sin 3A\cos A}{2\sin A\cos 3A} \quad ✓$$
$$= \tan 3A \cot A = \text{RHS}$$

(b) Prove that $\sin 5\theta + 2\sin 3\theta + \sin\theta = 4\sin 3\theta\cos^2\theta$.

$$\text{LHS} \equiv \sin 5\theta + 2\sin 3\theta + \sin\theta$$
$$\equiv (\sin 5\theta + \sin\theta) + 2\sin 3\theta \quad ✓$$
$$\equiv 2\sin\left(\frac{5\theta+\theta}{2}\right)\cos\left(\frac{5\theta-\theta}{2}\right) + 2\sin 3\theta \quad ✓$$
$$\equiv 2\sin 3\theta\cos 2\theta + 2\sin 3\theta$$
$$\equiv 2\sin 3\theta(\cos 2\theta + 1) \quad ✓$$
$$\equiv 2\sin 3\theta(2\cos^2\theta - 1 + 1) \quad ✓$$
$$\equiv 4\sin 3\theta\cos^2\theta \equiv \text{RHS}$$

(c) Prove that $\dfrac{\sin P + \cos(2Q-P)}{\cos P - \sin(2Q-P)} = \cot\left(\frac{\pi}{4} - Q\right)$

$$\text{LHS} \equiv \frac{\sin P + \cos(2Q-P)}{\cos P - \sin(2Q-P)}$$
$$\equiv \frac{\cos\left(\frac{\pi}{2} - P\right) + \cos(2Q-P)}{\sin\left(\frac{\pi}{2} - P\right) - \sin(2Q-P)} \quad ✓✓$$
$$\equiv \frac{2\cos\left(\frac{\frac{\pi}{2} - P + (2Q-P)}{2}\right)\cos\left(\frac{\frac{\pi}{2} - P - (2Q-P)}{2}\right)}{2\sin\left(\frac{\frac{\pi}{2} - P - (2Q-P)}{2}\right)\cos\left(\frac{\frac{\pi}{2} - P + (2Q-P)}{2}\right)} \quad ✓✓$$
$$\equiv \frac{\cos\left(\frac{\pi}{4} - Q\right)}{\sin\left(\frac{\pi}{4} - Q\right)} \quad ✓$$
$$\equiv \cot\left(\frac{\pi}{4} - Q\right) \equiv \text{RHS}$$

Calculator Free

5. [7 marks: 1, 3, 3]

(a) Prove that $\sin A\cos A = \frac{1}{2}\sin 2A$

$$\text{RHS} \equiv \frac{1}{2}\sin 2A$$
$$\equiv \frac{1}{2}(2\sin A\cos A) \quad ✓$$
$$\equiv \sin A\cos A \equiv \text{LHS}$$

(b) Prove that $\sin 40^\circ\cos 40^\circ\cos 80^\circ = \dfrac{\sin 20^\circ}{4}$.

$$\text{LHS} \equiv \sin 40^\circ\cos 40^\circ\cos 80^\circ$$
$$\equiv \frac{1}{2}\sin(2\times 40^\circ)\cos 80^\circ \equiv \frac{1}{2}\sin(80^\circ)\cos 80^\circ \quad ✓$$
$$\equiv \frac{1}{2}\times\left(\frac{1}{2}\sin(2\times 80^\circ)\right) \equiv \frac{1}{4}\sin 160^\circ \quad ✓$$
$$\equiv \frac{1}{4}\sin(180^\circ - 160^\circ) \quad ✓$$
$$\equiv \frac{1}{4}\sin 20^\circ \equiv \text{RHS}$$

(c) Use your answer in (b) to prove that $\cos 20^\circ\cos 40^\circ\cos 80^\circ = \frac{1}{8}$.

From (b):

$\sin 40^\circ\cos 40^\circ\cos 80^\circ = \frac{1}{4}\sin 20^\circ$ I

But $\sin 40^\circ = 2\sin 20^\circ\cos 20^\circ$. ✓

Hence, I becomes:

$(2\sin 20^\circ\cos 20^\circ)\times\cos 40^\circ\cos 80^\circ = \frac{1}{4}\sin 20^\circ$ ✓

$\Rightarrow \cos 20^\circ\cos 40^\circ\cos 80^\circ = \frac{1}{8}$ ✓

22 Trigonometric Identities V (Auxiliary Angles)

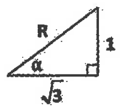
Calculator Assumed

1. [8 marks: 4, 4]

(a) Given that $\cos x + \sqrt{3}\sin x = R\sin(x+\alpha)$ where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$, use the formula for sin (A + B) to find in exact form the value of R and the exact value of α.

$R\sin(x+\alpha) = R\cos\alpha\sin x + R\sin\alpha\cos x$

Hence:
$R\cos\alpha = \sqrt{3}$
$R\sin\alpha = 1$ ✓
$\Rightarrow R = 2$ ✓
$\tan\alpha = \frac{1}{\sqrt{3}}$ ✓
$\alpha = \frac{\pi}{6}$ ✓



(b) Hence, find the maximum value (in exact form) for $y = \sqrt{3}\cos x + 3\sin x$ and the smallest positive value of x at which this occurs.

$y = \sqrt{3}\cos x + 3\sin x$
$= \sqrt{3}(\cos x + \sqrt{3}\sin x)$
$= \sqrt{3} \times 2\sin(x + \frac{\pi}{6})$ ✓

Hence, maximum value for $y = 2\sqrt{3}$ ✓
when $\sin(x + \frac{\pi}{6}) = 1$ ✓
$x + \frac{\pi}{6} = \frac{\pi}{2}$
$x = \frac{\pi}{3}$ ✓

Calculator Assumed

2. [10 marks: 4, 3, 3]

Compare $4\sin x + 7\cos x$ with the expansion of $R\sin(x+\alpha)$ where $0 \le \alpha \le 90°$. Hence, find the exact value of R and the value of α to 2 decimal places

Let $4\sin x + 7\cos x \equiv R\sin(x+\alpha)$
$\Rightarrow 4\sin x + 7\cos x \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

Compare coefficients for sin x and cos x:
$R\cos\alpha = 4$ (I)
$R\sin\alpha = 7$ (II) ✓
(II) ÷ (I) $\Rightarrow$ $\tan\alpha = \frac{7}{4}$ ✓
$\alpha = 60.26°$ ✓

$(I)^2 + (II)^2 \Rightarrow$ $R^2\cos^2\alpha + R^2\sin^2\alpha = 4^2 + 7^2$
$R^2(\cos^2\alpha + \sin^2\alpha) = 65$
$R = \sqrt{65}$ ✓

Hence, find:

(a) the maximum value (in exact form) for the expression $8\sin x + 14\cos x$ and the smallest positive value of x at which this occurs.

Since, $4\sin x + 7\cos x \equiv \sqrt{65}\sin(x + 60.26°)$,
$8\sin x + 14\cos x \equiv 2\sqrt{65}\sin(x + 60.26°)$
Maximum value for expression $= 2\sqrt{65}$. ✓

This occurs when $\sin(x + 60.26°) = 1$. ✓
$\Rightarrow x + 60.26° = 90°$
$x = 29.7°$ ✓

(b) the maximum value (in exact form) for the expression $-4\sin x - 7\cos x$ and the smallest positive value of x at which this occurs.

Since, $4\sin x + 7\cos x \equiv \sqrt{65}\sin(x + 60.26°)$,
$-4\sin x - 7\cos x \equiv -\sqrt{65}\sin(x + 60.26°)$
Maximum value for expression $= \sqrt{65}$. ✓

This occurs when $\sin(x + 60.26°) = -1$. ✓
$\Rightarrow x + 60.26° = 270°$
$x = 209.7°$ ✓

Calculator Assumed

3. [10 marks: 4, 3, 3]

Compare $5\cos x + 8\sin x$ with the expansion of form $R\cos(x - \alpha)$ where $0 \le \alpha \le \frac{\pi}{2}$. Hence, find the exact value of R and α to 4 decimal places.

Let $5\cos x + 8\sin x \equiv R\cos(x - \alpha)$
$\Rightarrow$ $5\cos x + 8\sin x \equiv R\cos x\cos\alpha + R\sin x\sin\alpha$

Compare coefficients for cos x and sin x:
$R\cos\alpha = 5$ (I)
$R\sin\alpha = 8$ (II) ✓
(II) ÷ (I) $\Rightarrow$ $\tan\alpha = \frac{8}{5}$ ✓
$\alpha = 1.0122$ ✓

$(I)^2 + (II)^2 \Rightarrow$ $R^2\cos^2\alpha + R^2\sin^2\alpha = 5^2 + 8^2$
$R^2(\cos^2\alpha + \sin^2\alpha) = 89$
$R = \sqrt{89}$ ✓
Hence, $5\cos x + 8\sin x \equiv \sqrt{89}\cos(x - 1.0122)$

Hence, find:

(a) the maximum value (in exact form) for the expression $\cos x + 1.6\sin x$ and the values of x where $0 \le x \le 2\pi$ at which this occurs.

Since, $5\cos x + 8\sin x \equiv \sqrt{89}\cos(x - 1.0122)$,
$\cos x + 1.6\sin x \equiv \frac{\sqrt{89}}{5}\cos(x - 1.0122)$

Maximum value for expression = $\frac{\sqrt{89}}{5}$. ✓

This occurs when $\cos(x - 1.0122) = 1$. ✓
$\Rightarrow$ $x - 1.0122 = 0$
$x = 1.01$ radians ✓

(b) the minimum value (in exact form) for the expression $10\sin x + 16\cos x$ and the values of x where $0 \le x \le 2\pi$ at which this occurs.

Since, $5\cos x + 8\sin x \equiv \sqrt{89}\cos(x - 1.0122)$, ✓
$10\cos x + 16\sin x \equiv 2\sqrt{89}\cos(x - 1.0122)$

Mminimum value for expression = $-2\sqrt{89}$.

This occurs when $\cos(x - 1.0122) = -1$. ✓
$\Rightarrow$ $x - 1.0122 = \pi$
$x = 4.15$ radians ✓

Calculator Assumed

4. [8 marks]

Use an appropriate trigonometric method to find the minimum value (in exact form) for $f(\theta) = 10 + 3\sin\theta + 5\cos\theta$ where $0 \le \theta \le 360^\circ$. Give also the smallest positive value for θ at which the minimum value of $f(\theta)$ occurs.

Let $3\sin\theta + 5\cos\theta \equiv R\sin(\theta + \alpha)$
$\Rightarrow$ $3\sin\theta + 5\cos\theta \equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

Compare coefficients for sin θ and cos θ:
$R\cos\alpha = 3$ (I)
$R\sin\alpha = 5$ (II) ✓
(II) ÷ (I) $\Rightarrow$ $\tan\alpha = \frac{5}{3}$ ✓
$\alpha = 59.04^\circ$ ✓

$(I)^2 + (II)^2 \Rightarrow$ $R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 5^2$
$R^2(\cos^2\alpha + \sin^2\alpha) = 34$
$R = \sqrt{34}$ ✓
Hence, $3\sin\theta + 5\cos\theta \equiv \sqrt{34}\sin(\theta + 59.04^\circ)$

Therefore $f(\theta) = 10 + 3\sin\theta + 5\cos\theta$ is identical to
$f(\theta) = 10 + \sqrt{34}\sin(\theta + 59.04^\circ)$. ✓

Minimum value for $f(\theta) = 10 - \sqrt{34}$. ✓

This occurs when $\sin(\theta + 59.04^\circ) = -1$. ✓
$\Rightarrow$ $\theta + 59.04^\circ = 270^\circ$
$\theta = 210.96^\circ$ ✓

23 Trigonometric Equations II

Calculator Free

1. [9 marks: 5, 4]

Solve for x within the given domain:

(a) $2\cos^2 x + 3\sin x = 0$ for $0 \le x \le 360^\circ$

$2(1-\sin^2 x) + 3\sin x = 0$ ✓
$2\sin^2 x - 3\sin x - 2 = 0$
$(2\sin x + 1)(\sin x - 2) = 0$ ✓
$\Rightarrow \sin x = -\frac{1}{2}$ or 2 (reject) ✓

$\sin x = -\frac{1}{2}$

Reference angle for $x = 30^\circ$.
Angle x is in Quadrant 3 or Quadrant 4.

Hence, $x = 180^\circ + 30^\circ, 360^\circ - 30^\circ$
$= 210^\circ, 330^\circ$ ✓✓

(b) $\cos x - 3\sec x - 2 = 0$ for $0 \le x \le 2\pi$

Rewrite as:

$$\cos x - \frac{3}{\cos x} - 2 = 0$$

Multiply both sides of equation with $\cos x$.
$\Rightarrow \cos^2 x - 2\cos x - 3 = 0$ ✓
$(\cos x + 1)(\cos x - 3) = 0$ ✓
$\Rightarrow \cos x = -1$ or 3 (reject) ✓

$\cos x = -1 \Rightarrow x = \pi$ radians ✓

Calculator Free

2. [13 marks: 3, 4, 6]

(a) Solve for x in $\cos x + \sqrt{3}\sin x = 0$ where $0 \le x \le 360^\circ$:

$\sqrt{3}\sin x = -\cos x$
$\tan x = -\frac{1}{\sqrt{3}}$ ✓

Reference angle for $x = 30^\circ$.
$\Rightarrow x = 150^\circ, 330^\circ$ ✓✓

(b) Solve for x in $\sin x - \cos 2x = 0$ where $-\pi < x \le \pi$.

$\sin x - (1 - 2\sin^2 x) = 0$ ✓
$2\sin^2 x + \sin x - 1 = 0$ ✓
$(2\sin x - 1)(\sin x + 1) = 0$
$\Rightarrow \sin x = \frac{1}{2}$ or -1

$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ ✓

$\sin x = -1 \Rightarrow x = -\frac{\pi}{2}$ ✓

(c) Find all values of x (in degrees) in $\cos x + \sin 2x = 0$.

$\cos x + 2\sin x\cos x = 0$ ✓
$\cos x(1 + 2\sin x) = 0$ ✓
$\Rightarrow \cos x = 0$ or $\sin x = -\frac{1}{2}$

$\cos x = 0 \Rightarrow \cos^{-1} x = 90^\circ$. ✓
$x = 360^\circ n \pm 90^\circ \quad n \in \mathbb{Z}$ ✓

$\sin x = -\frac{1}{2} \Rightarrow \sin^{-1} x = -30^\circ$. ✓
$x = -30^\circ \times (-1)^n + 180^\circ n \quad n \in \mathbb{Z}$ ✓

Calculator Free

3. [9 marks: 4, 5]

(a) Solve for θ in $\cos 2\theta + \cos\theta + 1 = 0$ for $0 \le \theta \le 2\pi$.

$(2\cos^2\theta - 1) + \cos\theta + 1 = 0$ ✓
$2\cos^2\theta + \cos\theta = 0$
$\cos\theta\,(2\cos\theta + 1) = 0$ ✓
$\Rightarrow \cos\theta = 0$ or $-\frac{1}{2}$

$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$ radians ✓

$\cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ radians ✓

(b) Find all solutions (in radians) to $\sin 2\theta - \sin\theta = 0$.

Rewrite as:
$2\sin\theta\cos\theta - \sin\theta = 0$. ✓
$\sin\theta\,(2\cos\theta - 1) = 0$ ✓
$\Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2}$ ✓

$\sin\theta = 0 \Rightarrow \theta = n\pi \quad n\in\mathbb{Z}$ ✓
$\cos\theta = \frac{1}{2} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3} \quad n\in\mathbb{Z}$ ✓

OR

$2\sin\left(\frac{2\theta-\theta}{2}\right)\cos\left(\frac{2\theta+\theta}{2}\right) = 0$ ✓
$\sin\frac{\theta}{2} = 0 \Rightarrow \frac{\theta}{2} = n\pi$ ✓
$\theta = 2n\pi \quad n\in\mathbb{Z}$ ✓
$\cos\frac{3\theta}{2} = 0 \Rightarrow \frac{3\theta}{2} = \frac{(2n+1)\pi}{2}$ ✓
$\theta = \frac{(2n+1)\pi}{3} \quad n\in\mathbb{Z}$ ✓

Calculator Free

4. [9 marks: 4, 5]

(a) Find all solutions (in radians) for θ in $3\tan^2\theta + 5\sec\theta + 1 = 0$.

$3(\sec^2\theta - 1) + 5\sec\theta + 1 = 0$ ✓
$3\sec^2\theta + 5\sec\theta - 2 = 0$
$(3\sec\theta - 1)(\sec\theta + 2) = 0$
$\Rightarrow \sec\theta = \frac{1}{3}$ or -2 ✓
$\sec\theta = \frac{1}{3} \Rightarrow \cos\theta = 3$ has no solution.
$\sec\theta = -2 \Rightarrow \cos\theta = -\frac{1}{2}$
$\theta = 2n\pi \pm \frac{2\pi}{3} \quad n\in\mathbb{Z}$ ✓✓

(b) Find all solutions (in degrees) for θ in $\tan\theta + \cot\theta - 2\sec\theta = 0$.

Rewrite as $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} - \frac{2}{\cos\theta} = 0.$ ✓
$\frac{\sin^2\theta + \cos^2\theta - 2\sin\theta}{\sin\theta\cos\theta} = 0$

$\Rightarrow \sin^2\theta + \cos^2\theta - 2\sin\theta = 0$ ✓
$1 - 2\sin\theta = 0$
$\Rightarrow \sin\theta = \frac{1}{2}$ ✓
$\sin^{-1}\theta = 30^\circ$ ✓
Hence: $\theta = (-1)^n \times 30^\circ + 180^\circ n \quad n\in\mathbb{Z}$ ✓

Calculator Free

5. [9 marks: 4, 5]

(a) Solve for all values of θ in $\tan(2\theta + \frac{\pi}{6}) = -1$.

$\tan^{-1}(2\theta + \frac{\pi}{6}) = -\frac{\pi}{4}$ ✓

$2\theta + \frac{\pi}{6} = n\pi - \frac{\pi}{4}$ ✓

$2\theta = n\pi - \frac{5\pi}{12}$ ✓

$\theta = \frac{n\pi}{2} - \frac{5\pi}{24} \quad n \in \mathbb{Z}$ ✓

(b) Solve for all values of θ in $2\cos 2\theta + 2\sin^2\theta - 9\cos\theta - 5 = 0$.

$2(2\cos^2\theta - 1) + 2(1 - \cos^2\theta) - 9\cos\theta - 5 = 0$ ✓✓

$2\cos^2\theta - 9\cos\theta - 5 = 0$ ✓

$(2\cos\theta + 1)(\cos\theta - 5) = 0$

$\cos\theta = -\frac{1}{2}$ or 5 (reject) ✓

$\theta = 2n\pi \pm \left(-\frac{2\pi}{3}\right)$

$= 2n\pi \pm \left(\frac{2\pi}{3}\right) \quad n \in \mathbb{Z}$ ✓

Calculator Free

6. [12 marks: 3, 5, 4]

(a) Solve for all values of θ in $\sqrt{3}\sin\theta + \cos\theta = 0$

Rewrite as: $\tan\theta = -\frac{1}{\sqrt{3}}$ ✓

$\theta = n\pi + \left(-\frac{\pi}{6}\right) \quad n \in \mathbb{Z}$ ✓✓

(b) Solve for all values of θ in $\sqrt{3}\sin\theta + \cos\theta = 1$

Rewrite: $\sqrt{3}\sin\theta + \cos\theta \equiv R\sin(\theta + \alpha)$

$R^2 = (\sqrt{3})^2 + 1^2$

$R = 2$

$\alpha = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

Hence: $\sqrt{3}\sin\theta + \cos\theta \equiv 2\sin(\theta + \frac{\pi}{6})$ ✓✓

$\Rightarrow 2\sin(\theta + \frac{\pi}{6}) = 1$

$\sin(\theta + \frac{\pi}{6}) = \frac{1}{2}$ ✓

$\theta + \frac{\pi}{6} = (-1)^n \times \frac{\pi}{6} \pm n\pi$ ✓

$\theta = (-1)^n \times \frac{\pi}{6} \pm n\pi - \frac{\pi}{6} \quad n \in \mathbb{Z}$ ✓

(c) Solve for all values of θ in $\sqrt{3}\sin\theta + \cos 2\theta = 1$.

Rewrite: $\sqrt{3}\sin\theta + (1 - 2\sin^2\theta) = 1$ ✓

$\sin\theta(\sqrt{3} - 2\sin\theta) = 0$ ✓

$\sin\theta = 0$ or $\frac{\sqrt{3}}{2}$

$\sin\theta = 0 \quad \Rightarrow \theta = n\pi \quad n \in \mathbb{Z}$ ✓

$\sin\theta = \frac{\sqrt{3}}{2} \quad \Rightarrow \theta = (-1)^n \times \frac{\pi}{3} \pm n\pi \quad n \in \mathbb{Z}$ ✓

Calculator Free

7. [11 marks: 5, 6]

(a) Solve for all values of θ in $\cos\theta + \sin\theta = 1$

Rewrite: $\cos\theta + \sin\theta \equiv R\cos(\theta - \alpha)$

$R^2 = 1^2 + 1^2$

$R = \sqrt{2}$

$\alpha = \tan^{-1} 1 = \frac{\pi}{4}$

Hence: $\sqrt{2}\cos(\theta - \frac{\pi}{4}) = 1$ ✓✓

$\cos(\theta - \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$(\theta - \frac{\pi}{4}) = 2n\pi \pm \cos^{-1}\frac{1}{\sqrt{2}}$ ✓

$= 2n\pi \pm \frac{\pi}{4}$

$\theta = 2n\pi$ or $2n\pi + \frac{\pi}{2}$ $n \in \mathbb{Z}$ ✓✓

(b) Solve for all values of θ in $1 + \sqrt{3}\tan\theta = \sqrt{3}\sec\theta$.

Rewrite: $1 + \frac{\sqrt{3}\sin\theta}{\cos\theta} = \frac{\sqrt{3}}{\cos\theta}$ where $\cos\theta \neq 0$.

$\cos\theta + \sqrt{3}\sin\theta = \sqrt{3}$ where $\cos\theta \neq 0$ ✓

Rewrite: $\cos\theta + \sqrt{3}\sin\theta \equiv R\cos(\theta - \alpha)$

$R^2 = 1^2 + 3$

$R = 2$

$\alpha = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

Hence: $2\cos(\theta - \frac{\pi}{3}) = \sqrt{3}$ ✓✓

$\cos(\theta - \frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

$(\theta - \frac{\pi}{3}) = 2n\pi \pm \cos^{-1}\frac{\sqrt{3}}{2}$ ✓

$= 2n\pi \pm \frac{\pi}{6}$

$\theta = 2n\pi + \frac{\pi}{6}$ or $2n\pi + \frac{\pi}{2}$

But $\cos\theta \neq 0$, hence, reject $\theta = 2n\pi + \frac{\pi}{2}$. ✓

Therefore: $\theta = 2n\pi + \frac{\pi}{6}$ $n \in \mathbb{Z}$ ✓

Calculator Free

8. [9 marks: 4, 5]

(a) Solve for all values of θ in $\cos\theta + \cos 3\theta = 0$.

$2\cos\left(\frac{3\theta + \theta}{2}\right)\cos\left(\frac{3\theta - \theta}{2}\right) = 0$ ✓

$\cos 2\theta = 0$ or $\cos\theta = 0$ ✓

For $\cos 2\theta = 0 \Rightarrow 2\theta = \frac{(2n+1)\pi}{2} \Rightarrow \theta = \frac{(2n+1)\pi}{4}$ $n \in \mathbb{Z}$ ✓

For $\cos\theta = 0 \Rightarrow \theta = \frac{(2n+1)\pi}{2}$ $n \in \mathbb{Z}$ ✓

OR

$\cos 3\theta = -\cos\theta$

$\cos 3\theta = \cos(\pi - \theta)$ ✓

$\Rightarrow 3\theta = 2n\pi \pm (\pi - \theta)$ ✓

$\theta = \frac{(2n+1)\pi}{4}$ or $\frac{(2n-1)\pi}{2}$ $n \in \mathbb{Z}$ ✓✓

(b) Solve $\cos\theta + \cos 3\theta + \cos 7\theta = 0$ for $0 \leq \theta \leq 180^\circ$.

Rewrite as: $\cos 7\theta + \cos\theta + \cos 3\theta = 0$

$2\cos\left(\frac{7\theta + \theta}{2}\right)\cos\left(\frac{7\theta - \theta}{2}\right) + \cos 3\theta = 0$ ✓

$2\cos 4\theta\cos 3\theta + \cos 3\theta = 0$

$\cos 3\theta\,(2\cos 4\theta + 1) = 0$ ✓

$\cos 3\theta = 0$ or $\cos 4\theta = -\frac{1}{2}$ ✓

For $\cos 3\theta = 0 \Rightarrow 3\theta = 90^\circ, 270^\circ, 450^\circ$

$\theta = 30^\circ, 90^\circ, 150^\circ$ ✓

For $\cos 4\theta = -\frac{1}{2} \Rightarrow 4\theta = 120^\circ, 240^\circ, 480^\circ$

$\theta = 30^\circ, 60^\circ, 120^\circ$ ✓

Calculator Assumed

9. [10 marks: 1, 3, 6]

(a) Show that $\sin 2\theta + \sin 3\theta \equiv 2\sin\frac{5\theta}{2}\cos\frac{\theta}{2}$.

$$\text{LHS} \equiv \sin 2\theta + \sin 3\theta$$
$$\equiv 2\sin\left(\frac{3\theta+2\theta}{2}\right)\cos\left(\frac{3\theta-2\theta}{2}\right) \quad ✓$$
$$\equiv 2\sin\frac{5\theta}{2}\cos\frac{\theta}{2} \equiv \text{RHS}$$

(b) Use your result in (a) to solve for all values of θ in $\sin 2\theta + \sin 3\theta = 0$

$$2\sin\left(\frac{3\theta+2\theta}{2}\right)\cos\left(\frac{3\theta-2\theta}{2}\right) = 0$$
$$\sin\frac{5\theta}{2} = 0 \text{ or } \cos\frac{\theta}{2} = 0 \quad ✓$$

For $\sin\frac{5\theta}{2} = 0 \Rightarrow \frac{5\theta}{2} = n\pi \Rightarrow \theta = \frac{2n\pi}{5} \quad n \in \mathbb{Z}$ ✓

For $\cos\frac{\theta}{2} = 0 \Rightarrow \theta = (2n+1)\pi \quad n \in \mathbb{Z}$ ✓

(c) Use your result in (a) to solve for all values of θ in $\sin 2\theta + \sin 3\theta - \sin\theta = 0$.

Rewrite: $2\sin\left(\frac{5\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \sin\theta = 0$

But $\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$.

Hence: $2\sin\left(\frac{5\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = 0$ ✓

$2\cos\left(\frac{\theta}{2}\right)[\sin\left(\frac{5\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)] = 0$ ✓

$2\cos\left(\frac{\theta}{2}\right) \times 2\sin\left(\frac{\frac{5\theta}{2}-\frac{\theta}{2}}{2}\right)\cos\left(\frac{\frac{5\theta}{2}+\frac{\theta}{2}}{2}\right) = 0$

$4\cos\left(\frac{\theta}{2}\right)\sin\theta\cos\left(\frac{3\theta}{2}\right) = 0$ ✓

$\sin\theta = 0 \Rightarrow \theta = 2n\pi \quad n \in \mathbb{Z}$ ✓

$\cos\left(\frac{\theta}{2}\right) = 0 \Rightarrow \theta = (2n+1)\pi \quad n \in \mathbb{Z}$ ✓

$\cos\left(\frac{3\theta}{2}\right) = 0 \Rightarrow \frac{3\theta}{2} = \frac{(2n+1)\pi}{2}$

$\theta = \frac{(2n+1)\pi}{3} \quad n \in \mathbb{Z}$ ✓

Calculator Assumed

10. [11 marks: 4, 2, 5]

(a) Use the formula for tan 2A to show that $\tan\frac{\pi}{8} = -1 + \sqrt{2}$.

$$\tan\left(2\times\frac{\pi}{8}\right) = \frac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)} \quad ✓$$
$$1-\tan^2\left(\frac{\pi}{8}\right) = 2\tan\left(\frac{\pi}{8}\right)$$
$$\tan^2\left(\frac{\pi}{8}\right) + 2\tan\left(\frac{\pi}{8}\right) - 1 = 0 \quad ✓$$
$$\tan\frac{\pi}{8} = \frac{-2\pm\sqrt{4+4}}{2} \quad ✓$$
$$= -1 \pm \sqrt{2}$$

But $\tan\frac{\pi}{8} > 0$, $\Rightarrow \tan\frac{\pi}{8} = -1 + \sqrt{2}$ ✓

(b) Use your answer in (a) to find all solutions to $\sqrt{2}\cos\theta - \cos\theta - \sin\theta = 0$.

Rewrite: $(\sqrt{2}-1)\cos\theta - \sin\theta = 0$

$\tan\theta = \sqrt{2} - 1$ ✓

$\theta = n\pi + \frac{\pi}{8}$ ✓

(c) Given that $\sin\frac{3\pi}{8} = \frac{1}{\sqrt{4-2\sqrt{2}}}$ and using the answer in (a),

solve for θ in $\sin\theta - (\sqrt{2}-1)\cos\theta = 1$ for $0 < \theta \le 2\pi$.

Rewrite: $\sin\theta - (\sqrt{2}-1)\cos\theta \equiv R\sin(\theta - \alpha)$

$R^2 = (\sqrt{2}-1)^2 + 1^2$

$R = \sqrt{4-2\sqrt{2}}$

$\alpha = \tan^{-1}(\sqrt{2}-1) = \frac{\pi}{8}$

Hence: $\sqrt{4-2\sqrt{2}}\sin(\theta - \frac{\pi}{8}) = 1$ ✓✓

$\sin(\theta - \frac{\pi}{8}) = \frac{1}{\sqrt{4-2\sqrt{2}}}$ ✓

Hence: $\theta - \frac{\pi}{8} = \frac{3\pi}{8}$ or $\frac{5\pi}{8}$ ✓

$\theta = \frac{\pi}{2}, \frac{3\pi}{4}$ ✓

24 Trigonometric Graphs

Calculator Free

1. [6 marks]

Complete the following table.

Function	Period	Amplitude	Phase Shift
$y = 2\sin(2x^{\circ})$	180°	2	0
$y = -4\cos(\frac{x}{2} + 30^{\circ})$	720°	4	-60°
$v = 10\tan(3t + \pi)$	$\frac{\pi}{3}$	n/a	$-\frac{\pi}{3}$
$Q = 5\sin(\frac{\pi}{2} - t)$	2π	5	$\frac{\pi}{2}$
$y = \frac{\sqrt{2}}{2}\cos(\pi t) + 100$	2	$\frac{\sqrt{2}}{2}$	0
$T = 5 - \sin(\frac{\pi}{4} - \theta)$	2π	1	$\frac{\pi}{4}$

[−1 mark per error]

2. [5 marks]

Complete the table below.

Function	Minimum value of function	Maximum value of function
$y = 3\sin t$	−3	3
$y = 20\cos(\frac{2x}{3} - 45^{\circ})$	−20	20
$v = 5\tan\theta$	n/a	n/a
$M = 2\sin(\frac{\pi}{2} - 3t) + 4$	$-2 + 4 = 2$	$2 + 4 = 6$
$y = 5 - \cos(2\pi t)$	$5 - 1 = 4$	$5 - (-1) = 6$

[−1 mark per error]

Calculator Free

3. [6 marks: 3, 3]

A trigonometric function has equation $y = -4\sin(2x + 30^{\circ})$ for $0^{\circ} \le x \le 360^{\circ}$. Use an algebraic method to find:

(a) the maximum value for y and the corresponding value(s) for x.

Maximum value for $y = -4 \times -1 = 4$. ✓

This occurs when $\sin(2x + 30^{\circ}) = -1$ ✓

$\Rightarrow \quad 2x + 30^{\circ} = 270^{\circ}$

$x = 120^{\circ}$ ✓

(b) the minimum value for y and the corresponding value(s) for x.

Minimum value for $y = -4 \times 1 = -4$. ✓

This occurs when $\sin(2x + 30^{\circ}) = 1$ ✓

$\Rightarrow \quad 2x + 30^{\circ} = 90^{\circ}$

$x = 30^{\circ}$ ✓

4. [5 marks]

A trigonometric function has equation $P = a\cos(bt + \frac{\pi}{4})$. Find the values of a (where $a > 0$) and b given that P has a maximum value of 4 when $t = \frac{\pi}{4}$.

$a = 4$ ✓

when $\cos(b \times \frac{\pi}{4} + \frac{\pi}{4}) = 1$. ✓

Hence, $\frac{b\pi}{4} + \frac{\pi}{4} = 2n\pi \quad n \in \mathbb{Z}$ ✓

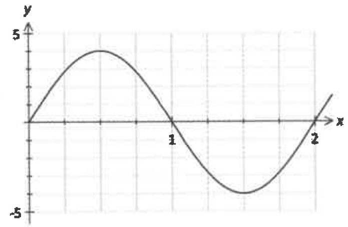
$\frac{b\pi}{4} = 2n\pi - \frac{\pi}{4}$

$b = 8n - 1 \quad n \in \mathbb{Z}$ ✓✓

Calculator Free

5. [3 marks]

The accompanying diagram shows the graph of a trigonometric function. State the amplitude and period of the function. Hence, give the equation of this function.



Amplitude = 4 ✓

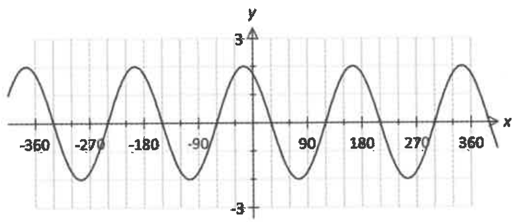
Period = 2 ✓

$f(x) = 4\sin \pi x$ ✓

6. [7 marks: 4, 3]

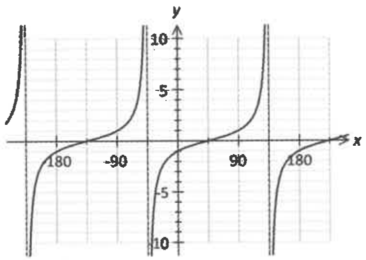
Find the equation of the following trigonometric functions:

(a)



Equation is $f(x) = 2\cos(2x + 30^{\circ})$ ✓✓✓✓

(b)

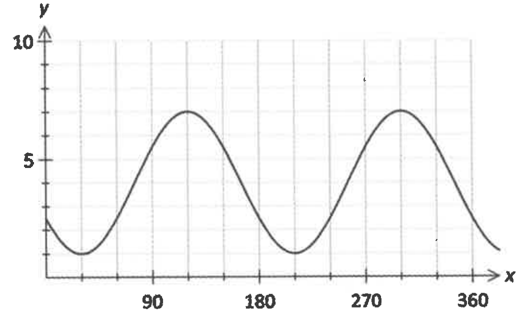


Equation is $y = \tan(x - 45^{\circ})$ ✓✓✓

Calculator Free

7. [4 marks]

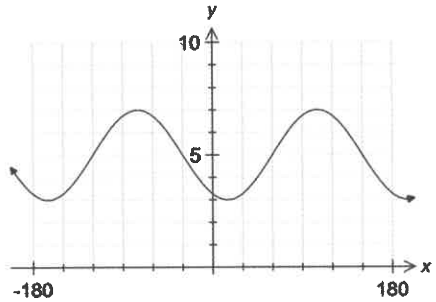
The graph of $y = a + b\sin(cx + d)$ is shown below.
Determine the values of a, b, c and d.



$a = 4, b = -3, c = 2, d = 30^{\circ}$ ✓✓✓✓

8. [4 marks]

The graph of $y = a + b\cos(cx + d)$ is shown below.
Determine the values of a, b, c and d.

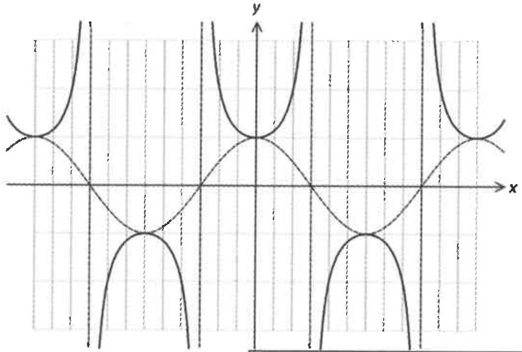


$a = 5$ $b = -2$ $c = 2$ $d = -30^{\circ}$ ✓✓✓✓

Calculator Free

9. [3 marks]

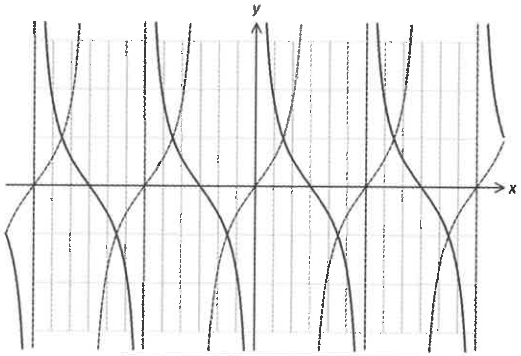
The graph of $y = \cos(ax)$ is shown below.
On the same diagram, sketch the graph of $y = \sec(ax)$.



Correct asymptotes.	✓
Correct minimum points.	✓
Correct shape/symmetry.	✓

10. [3 marks]

The graph of $y = \tan(ax)$ is shown below.
On the same diagram, sketch the graph of $y = \cot(ax)$.



Correct asymptotes.	✓
Correct roots.	✓
Correct points of intersection between curves.	✓

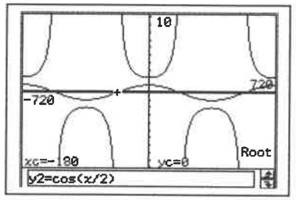
Calculator Assumed

11. [5 marks: 2, 1, 2]

Consider the curve with equation $y = 2\sec\left(\frac{x}{2}\right)$ for $-720^\circ < x < 720^\circ$.

(a) Determine the coordinates of the maximum turning point(s) of this curve.

$(-360^\circ, -2)$ & $(360^\circ, -2)$ ✓✓



(b) Determine the period of this curve.

Period = 720° ✓

(c) Determine the equations of the vertical asymptotes.

$x = \pm 540^\circ, \pm 180^\circ$ ✓✓

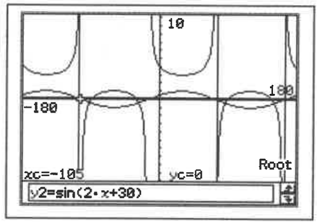
[Asymptotes coincide with the roots of $y = \cos\left(\frac{x}{2}\right)$.]

12. [5 marks: 2, 1, 2]

Consider the curve with equation $y = 2 + \operatorname{cosec}(2x + 30^\circ)$ for $-180^\circ < x < 180^\circ$.

(a) Determine the coordinates of the minimum turning point(s) of this curve.

$(-150^\circ, 3)$ & $(30^\circ, 3)$ ✓✓



(b) Determine the period of this curve.

Period = 180° ✓

(c) Determine the equations of the vertical asymptotes.

$x = -105^\circ, -15^\circ, 75^\circ, 165^\circ$ ✓✓

[Asymptotes coincide with the roots of $y = \sin(2x + 30^\circ)$.]

Calculator Assumed

13. [9 marks: 1, 1, 2, 2, 3]

The body temperature θ (Celsius) of a reptile in summer at time t hours after midnight is given by $\theta = 15 - 5\sin\left(\frac{\pi t}{12}\right)$.

(a) State the period for θ.

> Period = $\frac{2\pi}{\pi/12}$ = 24 hours ✓

(b) What is the range of body temperature experienced by the reptile?

> 10° Celsius to 20° Celsius. Hence, Range = 10°. ✓

(c) Find the minimum body temperature of the reptile and state when this first occurs after midnight.

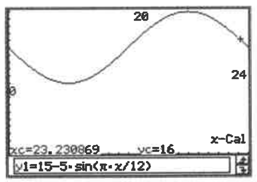
> Minimum body temperature is 10° Celsius. ✓
> This occurs when $t = 6$ i.e. at 6.00am. ✓

(d) Find the maximum body temperature of the reptile and state when this first occurs after midnight.

> Maximum temperature is 20° Celsius. ✓
> This occurs when $t = 18$ i.e. at 6.00 pm. ✓

(e) Find for how many hours in a 24 hour day, the body temperature of the reptile is below 16° Celsius. Give your answer to the nearest minute.

> From graph: $\theta \ge 16 \Rightarrow 12.7691 \le t \le 23.2309$. ✓
>
> That is, $\theta \ge 16$ for 10.4618 hours. ✓
>
> Hence, $\theta < 16$ for 13.5382 hours, i.e. for 13 hours and 32 minutes. ✓



Calculator Assumed

14. [10 marks: 2, 3, 5]

The water depth, h metres, measured from the bottom of a harbour, t hours after 6 am is modelled by the equation $h = 12 - 4\cos\left(\frac{\pi t}{6} - \frac{\pi}{4}\right)$ metres.

(a) Determine when the water depth is at its lowest in a 24-hour day (from 6 am). State the lowest depth.

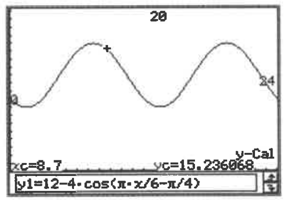
> Lowest depth = 8 metres ✓
> at $t = 1.5, 13.5$ i.e. at 7.30 am and 7.30 pm. ✓

(b) Repairs to the harbour can only be undertaken if the water depth is below 10 metres. What times of the day can this occur?

> From graph: $h < 10$ for $0 < t < 3.5$, $11.5 < t < 15.5$, $23.5 < t < 24$.
>
> That is: between 6.00 am and 9.30 am, ✓
> between 5.30 pm and 9.30 pm ✓
> between 5.30 am and 6.00 am of the next day. ✓

(c) The water depth is above k metres for 20% of a 24-hour day. Find k.

> $h > k$ for $24 \times 0.2 = 4.8$ hours. ✓
>
> The water depth changes with a period of 12 hours.
> There are two cycles within a 24-hour day. ✓
> Hence, within one cycle, the water depth is above k metres for 2.4 hours. ✓
>
> That is, $h > k$ for 1.2 hours before the maximum depth and for 1.2 hours after the maximum depth. ✓
>
> Maximum depth occurs at $t = 7.5$
> When $t = 7.5 + 1.2 = 8.7$, $h = 15.24$.
>
> Hence, $k = 15.24$ metres. ✓



25 Matrix Algebra

Calculator Free

1. [6 marks: 1, 1, 4] [TISC]

Consider the matrices $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) Given that matrix **X** can be added with matrix **A**, what is the dimension (size) of matrix **X** ?

dimension = 3 × 2 ✓

(b) Given that **BY** = **YB**, what is the dimension (size) of matrix **Y** ?

dimension = 2 × 2 ✓

(c) Consider the matrices, $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ k \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \end{pmatrix}$.

Two different matrices are selected from the three given and then multiplied together. State all the possible products.

$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} -1 \\ 1+k \end{pmatrix}$ ✓

$\begin{pmatrix} 1 \\ k \end{pmatrix}\begin{pmatrix} -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -k & k \end{pmatrix}$ ✓

$\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = (-1+k)$ ✓

$\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix}$ ✓

Calculator Free

2. [5 marks: 1, 1, 1, 1, 1] [TISC]

$\mathbf{A} = \begin{pmatrix} -27 & -3 \\ 1 & 5 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 4 & -3 \\ 8 & 5 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} -15 & -12 \\ 25 & 20 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} -15 & 12 \\ 25 & 20 \end{pmatrix}$

$\mathbf{E} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$ $\mathbf{F} = \begin{pmatrix} -27 \\ 1 \end{pmatrix}$ $\mathbf{G} = \begin{pmatrix} 5 & 4 \end{pmatrix}$ $\mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Choose a matrix from the list above to make the following statements true. Write the name of the matrix in the space provided.

(a) .. C ✓ is a square singular matrix.

(b) **E** × **G** = .. C ✓ ..

(c) **D** × **I** = .. D ✓ ...

(d) C ✓ − 3**B** = **A**

(e) **B** × E ✓ = **F**

3. [5 marks: 2, 3] [TISC]

Let $\mathbf{A} = \begin{pmatrix} k & 1 \\ 8 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ -8 & k \end{pmatrix}$

(a) Find the value(s) of k if **A** is a non-singular matrix.

$|\mathbf{A}| = 2k - 8 \neq 0$ ✓
$k \neq 4$ ✓

(b) Find the value(s) of k if **A** × **B** = **A** + **B**.

$\begin{pmatrix} 2k-8 & 0 \\ 0 & 2k-8 \end{pmatrix} = \begin{pmatrix} k+2 & 0 \\ 0 & k+2 \end{pmatrix}$ ✓✓

$\Rightarrow 2k - 8 = k + 2$

$k = 10$ ✓

Calculator Assumed

4. [5 marks: 1, 2, 2] [TISC]

Given that $\mathbf{A} = \begin{pmatrix} k-1 & 0 \\ 0 & k+1 \end{pmatrix}$.

(a) Find in terms of k, the determinant of matrix **A**.

> $(k-1)(k+1) = k^2 - 1$ ✓

(b) Find the value(s) of k for which **A** is singular.

> $k = -1$ or 1 ✓✓

(c) Given that A is non-singular, find $\mathbf{A}^{-1}$ in terms of k.

> $\mathbf{A}^{-1} = \dfrac{1}{k^2-1}\begin{pmatrix} k+1 & 0 \\ 0 & k-1 \end{pmatrix}$ ✓ ✓

---

5. [6 marks: 2, 2, 2] [TISC]

Given that $\mathbf{A} = \begin{pmatrix} a & 1 \\ b & a \end{pmatrix}$.

(a) Find the relationship between a and b such that **A** is a singular matrix.

> As **A** is a singular matrix, $|\mathbf{A}| = 0$. ✓
> $|\mathbf{A}| = a^2 - b = 0$ ✓

(b) Given that $b = 4$, find the value(s) of a for which **A** is non-singular.

> Since A is non-singular, $|\mathbf{A}| \neq 0$.
> Hence, $a^2 - 4 \neq 0$ ✓
> $\Rightarrow$ $a \neq \pm 2$ ✓ (must give both values)

(c) Explain clearly why **A** will always have an inverse if $b < 0$.

> If $b < 0$, then $a^2 - b > 0 \;\; \forall a$. ✓
> Hence, $|\mathbf{A}| \neq 0$. ✓
> $\Rightarrow$ **A** will always have an inverse.

Calculator Free

6. [6 marks: 1, 2, 3]

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -1 & 5 \\ 5 & -2 \end{pmatrix}$.

(a) Find $\mathbf{B}^{-1}$.

> $\mathbf{B}^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. ✓

(b) Find $\mathbf{C} \times \mathbf{B}^{-1}$.

> $\mathbf{CB}^{-1} = \begin{pmatrix} -1 & 5 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.
> $= \begin{pmatrix} -1 & 3 \\ 5 & 8 \end{pmatrix}$ ✓✓

(c) Find **X** if $(\mathbf{A} + \mathbf{X})\,\mathbf{B} = \mathbf{C}$.

> $(\mathbf{A} + \mathbf{X}) = \mathbf{C}\,\mathbf{B}^{-1}$
> $= \begin{pmatrix} -1 & 3 \\ 5 & 8 \end{pmatrix}$ ✓
> $\mathbf{X} = \begin{pmatrix} -1 & 3 \\ 5 & 8 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ ✓
> $= \begin{pmatrix} -2 & 1 \\ 4 & 5 \end{pmatrix}$ ✓

Calculator Free

7. [4 marks: 2, 2] [TISC]

(a) If **A** is a non-singular square matrix, show that if $\mathbf{A}^2 = \mathbf{A}$, then $\mathbf{A} = \mathbf{I}$ where **I** is the appropriate identity matrix.

$$\mathbf{A}^2 = \mathbf{A}$$
$$\mathbf{A}^2\mathbf{A}^{-1} = \mathbf{A}\mathbf{A}^{-1} \quad ✓$$
$$\mathbf{A} = \mathbf{I} \quad ✓$$

(b) Find a 2 × 2 non-zero matrix **A**, where $\mathbf{A}^2 = \mathbf{A}$ and $\mathbf{A} \neq \mathbf{I}$. (**I** is the 2 × 2 identity matrix.)

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad ✓✓$$

---

8. [6 marks: 3, 3]

(a) Given that **P** and **Q** are square matrices and $\mathbf{PQ} = \mathbf{P} + \mathbf{Q}$, show that $\mathbf{P} = \mathbf{Q}(\mathbf{Q} - \mathbf{I})^{-1}$, where **I** is the appropriate identity matrix.

$$\mathbf{PQ} - \mathbf{P} = \mathbf{Q} \quad ✓$$
$$\mathbf{P}(\mathbf{Q} - \mathbf{I}) = \mathbf{Q} \quad ✓$$
$$\mathbf{P} = \mathbf{Q}(\mathbf{Q} - \mathbf{I})^{-1} \quad ✓$$

(b) Given that **A** and **B** are 2 × 2 non-zero diagonal matrices, prove that **A** and **B** are commutative under multiplication.

Let $\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$

$$\mathbf{AB} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} ax & 0 \\ 0 & by \end{pmatrix} \quad ✓$$

$$\mathbf{BA} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} ax & 0 \\ 0 & by \end{pmatrix} \quad ✓$$

Hence, **AB** = **BA**.
That is, **A** and **B** are commutative under multiplication. ✓

Calculator Free

9. [9 marks: 2, 3, 4]

Given that the non-singular matrix **A**, where $\mathbf{A}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{A}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

find each of the following. Justify each of your answers.

(a) $\mathbf{A}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$$\mathbf{A}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \mathbf{A}\left[\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$
✓ ✓

(b) $\mathbf{A}^2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$$\mathbf{A}\left[\mathbf{A}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right] = \mathbf{A}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$
✓ ✓ ✓

(c) $\mathbf{A}^{-1}\begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 2 \\ 2 & -1 & 1 \end{pmatrix}$

$$\mathbf{A}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \Rightarrow \mathbf{A}^{-1}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad ✓$$

$$\mathbf{A}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \Rightarrow \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad ✓$$

$$\mathbf{A}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \mathbf{A}^{-1}\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad ✓$$

$$\mathbf{A}^{-1}\begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 2 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix} \quad ✓$$

Calculator Assumed

10. [5 marks]

Given that $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with all non-zero elements, where $|\mathbf{M}| = 1$ and $\mathbf{M}^{-1} = \mathbf{M}^2$, prove that $a + d = -1$.

$$\mathbf{M}^{-1} = \mathbf{M}^2 \Rightarrow \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+bc \end{pmatrix}$$

$|\mathbf{M}| = 1 \Rightarrow ad - bc = 1$

Hence $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+bc \end{pmatrix}$ ✓✓✓

$\Rightarrow a^2 + bc = d$ I $\quad ab + bd = -b$ II

$ac + cd = -c$ III $\quad d^2 + bc = a$ IV

From II: $b(a + d) = -b \Rightarrow a + d = -1$ ✓

From III: $c(a + d) = -c \Rightarrow a + d = -1$ ✓

Hence: $a + d = -1$

---

11. [4 marks]

Let **X** be a $n \times 1$ matrix, **A** be a $n \times n$ matrix and λ be a real non-zero constant. Given that $\mathbf{A}\,\mathbf{X} = \lambda\mathbf{X}$, prove that $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

$\mathbf{A}\mathbf{X} = \lambda\mathbf{X} \Rightarrow \mathbf{A}\mathbf{X} - \lambda\mathbf{X} = \mathbf{0}$

$(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = \mathbf{0}$ ✓

$|(\mathbf{A} - \lambda\mathbf{I})\mathbf{X}| = |\mathbf{0}|$ ✓

$|(\mathbf{A} - \lambda\mathbf{I})| \times |\mathbf{X}| = 0$ ✓

But $|\mathbf{X}|$ is not defined.

$\Rightarrow |(\mathbf{A} - \lambda\mathbf{I})| = 0$ ✓

Calculator Assumed

12. [7 marks: 2, 2, 3] [TISC]

Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}$.

(a) Show that $\mathbf{A}^2 = \mathbf{A} + k\mathbf{I}$ where k is a real constant.

$$\mathbf{A}^2 = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}^2 = \begin{pmatrix} 9 & 1 \\ 5 & 6 \end{pmatrix}$$

$\begin{pmatrix} 9 & 1 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix} + k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$k = 7$ ✓

(b) Use your result in (a) to find $\mathbf{A}^{-1}$ in the form $\alpha\mathbf{A} + \beta\mathbf{I}$.

$\mathbf{A}^2 = \mathbf{A} + 7\mathbf{I}$

$\mathbf{A}^2 - \mathbf{A} = 7\mathbf{I}$

$\mathbf{A}(\mathbf{A} - \mathbf{I}) = 7\mathbf{I}$ ✓

$\mathbf{A}^{-1} = \frac{1}{7}\mathbf{A} - \frac{1}{7}\mathbf{I}$. ✓

(c) Find $\mathbf{A}^4$ in terms of **A** and **I**.

$\mathbf{A}^2 = \mathbf{A} + 7\mathbf{I}$

$\mathbf{A}^4 = [\mathbf{A} + 7\mathbf{I}]^2$ ✓

$= \mathbf{A}^2 + 14\mathbf{A} + 49\mathbf{I}$

$= \mathbf{A} + 7\mathbf{I} + 14\mathbf{A} + 49\mathbf{I}$ ✓

$= 15\mathbf{A} + 56\mathbf{I}$ ✓

Calculator Assumed

13. [8 marks: 3, 1, 4]

(a) Given that **A**, and **B** are non-singular square matrices prove that

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

Let $(\mathbf{AB})^{-1} = \mathbf{P}$.

$\Rightarrow \quad (\mathbf{AB}) \times \mathbf{P} = \mathbf{I}$

$\mathbf{A}^{-1}\mathbf{AB} \times \mathbf{P} = \mathbf{A}^{-1}$ ✓

$\mathbf{B} \times \mathbf{P} = \mathbf{A}^{-1}$ ✓

$\mathbf{B}^{-1}\mathbf{B} \times \mathbf{P} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

$\mathbf{P} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ ✓

(b) Hence, or otherwise, prove that $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$.

$(\mathbf{AA})^{-1} = \mathbf{A}^{-1}\mathbf{A}^{-1}$ ✓

$(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$.

(c) Given that $\mathbf{A}^2 = 2\mathbf{A} + \mathbf{I}$, find $(\mathbf{A}^2)^{-1}$ in the form $p\mathbf{A} + q\mathbf{I}$ where p and q are real constants and **I** is the identity matrix.

$\mathbf{A}^{-1}\mathbf{A}^2 = \mathbf{A}^{-1}(2\mathbf{A} + \mathbf{I})$ ✓

$\mathbf{A} = 2\mathbf{I} + \mathbf{A}^{-1}$

$\mathbf{A}^{-1} = \mathbf{A} - 2\mathbf{I}$ ✓

From (b) $\quad (\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$

$= (\mathbf{A} - 2\mathbf{I})(\mathbf{A} - 2\mathbf{I})$ ✓

$= \mathbf{A}^2 - 4\mathbf{A} + 4\mathbf{I}$

$= (2\mathbf{A} + \mathbf{I}) - 4\mathbf{A} + 4\mathbf{I}$

$= -2\mathbf{A} + 5\mathbf{I}$ ✓

26 Systems of Equations

Calculator Free

1. [6 marks: 1, 1, 2, 2]

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$.

(a) Find $\mathbf{A}^{-1}$.

$\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}^{-1} = -\frac{1}{5}\begin{pmatrix} -3 & -2 \\ -1 & 1 \end{pmatrix}$ ✓

(b) Find the product $\mathbf{A}^{-1}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

$\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ ✓

(c) Consider the system of equations:

$$x + 2y = 2$$
$$-x + 3y = -2$$

(i) Rewrite the given system of equations in the form $\mathbf{AX} = \mathbf{B}$ where **X** is a column matrix and **A** and **B** are appropriate matrices.

Rewrite system as: $\quad x + 2y = 2$

$x - 3y = 2$ ✓

Hence: $\quad \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ ✓

(ii) Use a matrix method to solve for x and y.

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ ✓

$= \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Hence, $x = 2$, $y = 0$ ✓

Calculator Assumed

2. [6 marks: 1, 1, 2, 2]

Let $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$.

(a) Find $\mathbf{A}^{-1}$.

$$\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix} \quad ✓$$

(b) Find the product $(3 \quad 4) \times \mathbf{A}^{-1}$.

$$(3 \quad 4) \times \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}^{-1} = (3 \quad 4) \times \frac{1}{2}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}$$
$$= (-4 \quad 3) \quad ✓$$

(c) Consider the system of equations:

$$3x + 5y = 3$$
$$x + 2y = 2$$

(i) Rewrite the given system of equations in the form **XA** = **B** where **X** is a row matrix and **A** and **B** are appropriate matrices.

Rewrite system as: $3x + 5y = 3$, $2x + 4y = 4$ ✓

Hence: $(x \quad y)\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} = (3 \quad 4)$ ✓

(ii) Use a matrix method to solve for x and y.

$$(x \quad y) = (3 \quad 4) \times \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}^{-1} \quad ✓$$
$$= (-4 \quad 3)$$

Hence: $x = -4, y = 3$ ✓

Calculator Free

3. [4 marks]

Given that $\mathbf{A} \times \mathbf{B} = 4\mathbf{I}$, $\mathbf{B} \times \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{A} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, where **I** is the 2 × 2 unit matrix, find x and y.

$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ ✓

But $\mathbf{A} \times \mathbf{B} = 4\mathbf{I} \Rightarrow \mathbf{A}^{-1} = \frac{1}{4} \times \mathbf{B}$ ✓

Hence, $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4} \times \mathbf{B} \times \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

$= \frac{1}{4} \times \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. ✓

$x = \frac{1}{4}, y = \frac{1}{2}$ ✓

4. [5 marks: 1, 4]

Let $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$.

(a) Find product **A** × **B**.

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 13 \\ 0 & 1 \end{pmatrix} \quad ✓$$

(b) Show how your answer in (a) can be used to solve for x and y in:
$3x + 5y = 1$ and $x + 2y = 2$.

Rewrite as: $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Pre-multiply both sides with $\begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad ✓$$
$$\begin{pmatrix} 7 & 13 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \end{pmatrix} \quad ✓$$

Hence: $y = 5$ ✓

$7x + 13(5) = 9 \Rightarrow x = -8$ ✓

27 Applications using Matrices

Calculator Assumed

1. [7 marks: 2, 3, 2]

A budget airline flies from Perth to Sydney and charges two different fares for its passengers; deluxe economy and economy. On a certain flight, there were: 200 fare paying passengers and one and a half times as many economy passengers as deluxe economy passengers
Let d: number of deluxe economy passengers on this flight
e: number of economy passengers on this flight

(a) Use the information given above to write down two equations involving d and e.

$d + e = 200$ ✓
$e - 1.5d = 0$ ✓

(b) Use a method involving the inverse of a matrix to find d and e.

$$\begin{pmatrix} 1 & 1 \\ -1.5 & 1 \end{pmatrix}\begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} 200 \\ 0 \end{pmatrix}$$ ✓

$$\begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1.5 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 200 \\ 0 \end{pmatrix}$$ ✓

$$= \begin{pmatrix} 80 \\ 120 \end{pmatrix}$$

Hence: $d = 80, e = 120$ ✓

(c) A deluxe economy ticket costs $349, while an economy ticket costs 30% less. Use a matrix method to find the total amount in fares collected from this flight. You need to show clearly the matrices used, and the operation(s) used on these matrices.

$$\begin{pmatrix} 349 & 349 \times 0.7 \end{pmatrix}\begin{pmatrix} 80 \\ 120 \end{pmatrix} = \begin{pmatrix} 57\,236 \end{pmatrix}$$ ✓

Hence, total fare collected = $57 236 ✓

Calculator Assumed

2. [5 marks: 1, 1, 3]

The table below shows the number of hours Jack worked last week at a fast food outlet.

Shift	Weekdays	Weekends
Morning (M)	16	4
Afternoon (A)	8	4
Night (N)	8	0

(a) Write a *row* matrix **A** describing the number of hours Jack worked on each shift on weekdays.

M A N
A = Weekday $\begin{pmatrix} 16 & 8 & 8 \end{pmatrix}$ ✓

(b) Write a *column* matrix **B** describing the number of hours Jack worked on each shift on weekends.

Weekend
B = M, A, N $\begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$ ✓

The rates of pay are $15.00 per hour for weekday morning shifts, $12.00 per hour for weekday afternoon shifts and $20 per hour for weekday night shifts. Jack is paid twice as much per hour for weekend shifts as weekday shifts.

(c) Use matrices **A** and **B** and other matrices as required to find the total amount of money Jack earned last week.

$$\begin{pmatrix} 16 & 8 & 8 \end{pmatrix}\begin{pmatrix} 15 \\ 12 \\ 20 \end{pmatrix} + \begin{pmatrix} 30 & 24 & 40 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$$ ✓✓

$= (496) + (216)$
$= (712)$

Amount of money earned = $712. ✓

Calculator Assumed

3. [7 marks: 2, 2, 3]

The number of different tickets available for a charity concert at a concert hall is given in the table below.

	Stalls	Gallery
Adults	500	300
Students/pensioners	150	50

The prices for these tickets are given in the table below.

	Stalls	Gallery
Adults	\$150	\$90
Students/pensioners	\$120	\$70

(a) Given that all the tickets were sold, use a matrix method to determine the revenue from:
(i) the adult members of audience

$$\begin{pmatrix}150 & 90\end{pmatrix}\begin{pmatrix}500\\300\end{pmatrix} = \begin{pmatrix}102\,000\end{pmatrix}$$ ✓

Hence, \$102 000. ✓
(No marks if matrices are not used)

(ii) the stalls tickets.

$$\begin{pmatrix}150 & 120\end{pmatrix}\begin{pmatrix}500\\150\end{pmatrix} = \begin{pmatrix}93\,000\end{pmatrix}$$ ✓

Hence, \$93 000. ✓
(No marks if matrices are not used)

(b) Given that 310 gallery tickets were sold realising a revenue of \$26 900. Use a matrix method to determine how many of the gallery tickets were sold to adults?

Let a: number of gallery tickets sold to adults
p: number of gallery tickets sold to pensioners/students.
Then: $a + p = 310$
$90a + 70p = 26\,900$

Hence: $$\begin{pmatrix}1 & 1\\90 & 70\end{pmatrix}\begin{pmatrix}a\\p\end{pmatrix} = \begin{pmatrix}310\\26\,900\end{pmatrix}$$ ✓

$$\begin{pmatrix}a\\p\end{pmatrix} = \begin{pmatrix}1 & 1\\90 & 70\end{pmatrix}^{-1}\begin{pmatrix}310\\26\,900\end{pmatrix}$$ ✓

$$= \begin{pmatrix}260\\50\end{pmatrix}$$

Hence, 260 tickets were sold to adults. ✓

Calculator Assumed

4. [8 marks: 2, 3, 3]

In a city there are two companies, A and B, that supply gas to households. The table below shows the percentage of customers in each company that will remain with their original supplier and the percentage of customers that will switch to the competitor within two years.

%		From A	From B
To	A	65	15
	B	35	85

Initially, there were 750 000 and 250 000 customers with companies A and B respectively. Assume that the total number of customers remain unchanged.

(a) Use a matrix method to determine the number of customers with company A at the end of two years.

$$\begin{pmatrix}0.65 & 0.15\end{pmatrix}\begin{pmatrix}750\,000\\250\,000\end{pmatrix} = \begin{pmatrix}525\,000\end{pmatrix}$$ ✓

Hence, 525 000 customers with A after 2 years. ✓

(b) Use a matrix method to determine the number of customers with company A at the end of four years.

After two years, there will be 525 000 customers with A and 475 000 customers with B. ✓

Hence, after 4 years: $$\begin{pmatrix}0.65 & 0.15\end{pmatrix}\begin{pmatrix}525\,000\\475\,000\end{pmatrix} = \begin{pmatrix}412\,500\end{pmatrix}$$ ✓

That is, there will be 412 500 customers with A. ✓

(c) Using the table given, will company B ever have 750 000 customers? Justify your answer.

Distribution of customers after n lots of 2 years is given by:

$$\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0.65 & 0.15\\0.35 & 0.85\end{pmatrix}^n\begin{pmatrix}750\,000\\250\,000\end{pmatrix}$$ ✓

For $n = 30$: $$\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}300\,000\\700\,000\end{pmatrix}$$

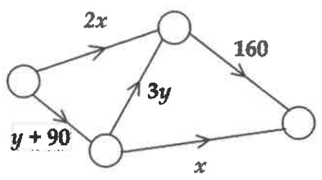
For $n = 31$: $$\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}300\,000\\700\,000\end{pmatrix}$$

Hence, no, as the maximum number of customers for B is 700 000. ✓✓

Calculator Assumed

5. [7 marks: 3, 4]

The diagram below shows the flow of fluid (in litres/minute) through a network of pipes. The numbers or letters indicate the flow rate through the pipe concerned. Assume that no fluid is lost in the process.



(a) Write down all equations involving x and y for the given network.

$2x + 3y = 160$ (1) ✓

$y + 90 = x + 3y$
$\Rightarrow x + 2y = 90$ (2) ✓

$2x + y + 90 = 160 + x$
$x + y = 70$ (3) ✓

(b) Use a method involving the inverse of a matrix to solve for x and y. Show clearly the matrices involved.

Using (1) and (2):

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 160 \\ 90 \end{pmatrix}$$ ✓

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 160 \\ 90 \end{pmatrix}$$ ✓

$$= \begin{pmatrix} 50 \\ 20 \end{pmatrix}$$

Hence: $x = 50, y = 20$ ✓

Check: Clearly $x = 50, y = 20$ satisfies (3). ✓

Calculator Assumed

6. [7 marks: 3, 4]

Mr Smart uses the following encryption procedure to code a four digit PIN.

- Replace each digit x, using the function $f(x) = (x + 10)^2$.
  For example, the PIN 5678 becomes $f(5)\ f(6)\ f(7)\ f(8)$, i.e. 225 256 289 324.
- The replacement is then multiplied by $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$. Hence,

5678 $\Rightarrow$ 225 256 289 324 $\Rightarrow$ 838 805 770 869

(a) Use Mr Smart's procedure to encrypt the PIN 9517. Show clearly each stage of the process.

9517 $\Rightarrow f(9)f(5)f(1)f(7) \Rightarrow$ 361 225 121 289 ✓

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 361 \\ 225 \\ 121 \\ 289 \end{pmatrix} = \begin{pmatrix} 771 \\ 875 \\ 707 \\ 635 \end{pmatrix}$$ ✓

Hence, 771 875 707 635 ✓

(b) Mr Smart's partner Ninety9, receives the sequence of numbers 813 621 726 786. Decrypt this sequence of numbers to determine Mr Smart's PIN. Show clearly how you obtained your answer.

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 813 \\ 621 \\ 726 \\ 786 \end{pmatrix} = \begin{pmatrix} 196 \\ 169 \\ 361 \\ 256 \end{pmatrix}$$ ✓

Hence, PIN is $f^{-1}(196)f^{-1}(169)f^{-1}(361)f^{-1}(256)$ ✓

$f(x) = (x + 10)^2 \Rightarrow f^{-1}(x) = -10 + \sqrt{x}$, since digits ≥ 0. ✓

Therefore, PIN is 4396. ✓

Calculator Assumed

7. [8 marks: 3, 3, 2]

The matrix $\mathbf{T} = \begin{matrix} & \text{From} \\ \text{To} & \end{matrix} \begin{matrix} & A & B & C & D \\ A & 0.75 & 0.25 & 0.45 & 0.2 \\ B & 0.15 & 0.45 & 0.25 & 0.1 \\ C & 0.03 & 0.15 & 0.25 & 0.05 \\ D & 0.07 & 0.15 & 0.05 & 0.65 \end{matrix}$ show the percentages of

customers from each of the Internet Service Providers A, B, C and D that remain with their original provider and those that will switch providers after 1 year. For example, 75% of customers with A will remain with A after one year while 15% will leave A to join B, 3% will leave A to join C and 7% will leave A to join D.

John starts with B. This may be represented by the column matrix $\mathbf{X} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

(a) Use matrices **T** and **X** to find the probability that John will still be with B:

(i) after two years.

$$\begin{pmatrix} 0.75 & 0.25 & 0.45 & 0.2 \\ 0.15 & 0.45 & 0.25 & 0.1 \\ 0.03 & 0.15 & 0.25 & 0.05 \\ 0.07 & 0.15 & 0.05 & 0.65 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.45 \\ 0.15 \\ 0.15 \end{pmatrix} \quad ✓✓$$

Hence, required probability = 0.45. ✓

(ii) after 10 years

$$\begin{pmatrix} 0.75 & 0.25 & 0.45 & 0.2 \\ 0.15 & 0.45 & 0.25 & 0.1 \\ 0.03 & 0.15 & 0.25 & 0.05 \\ 0.07 & 0.15 & 0.05 & 0.65 \end{pmatrix}^9 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.51 \\ 0.21 \\ 0.08 \\ 0.20 \end{pmatrix} \quad ✓✓$$

Hence, required probability is 0.21. ✓

(b) Which will be the dominant provider after 10 years. Justify your answer.

$$T^9 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.51 \\ 0.21 \\ 0.08 \\ 0.20 \end{pmatrix}, \ T^9 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.51 \\ 0.21 \\ 0.08 \\ 0.20 \end{pmatrix}, \ T^9 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.51 \\ 0.21 \\ 0.08 \\ 0.20 \end{pmatrix}$$

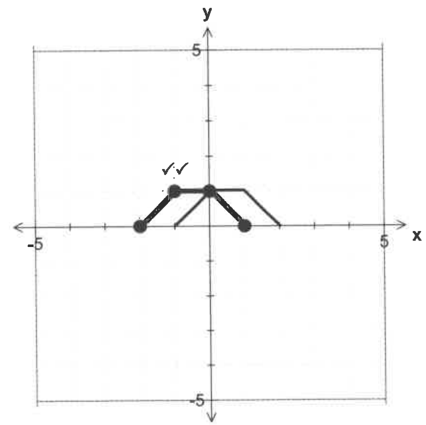
After 10 years, there is a 51% chance that A will pick up customers who started off with any other service provider.
Hence, A will be the dominant provider. ✓✓

28 Transformation Matrices

Calculator Free

1. [6 marks: 2, 2, 2]

The graph of $y = f(x)$ is given below.



The transformation represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is applied to this curve.

(a) Find the image of the points (–1, 0), (0, 1), (1, 1) and (2, 0).

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad ✓$$

Hence, images are respectively (1, 0), (0, 1), (–1, 1) and (–2, 0). ✓

(b) Sketch on the axes provided above, the graph of the resulting curve.

(c) The equation of the resulting curve is $y = a\,f(bx + c)$. Find a, b and c.

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ represents a reflection about the y-axis. ✓

Hence $y = f(x)$ is transformed to $y = f(-x)$.
$\Rightarrow a = 1, b = -1, c = 0.$ ✓

Calculator Free

2. [9 marks: 2, 3, 2, 2] [TISC]

Consider the curve with equation $y = f(x)$. The curve has a maximum point at A $(-1, 3)$ and a minimum point at B $(4, -7)$. The curve $y = f(x)$ is mapped onto the curve $y = g(x)$ by a transformation represented by the matrix $\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(a) Describe the effect the transformation represented by T has on the graph of $y = f(x)$.

> The transformation reflects the graph of $y = f(x)$ about the x-axis. ✓ ✓

(b) Find the coordinates of the images of the points A and B under **T**.

> $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 3 & -7 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -3 & 7 \end{pmatrix}$ ✓
> Hence, image of A is $(-1, -3)$. ✓
> image of B is $(4, 7)$. ✓

(c) Find the coordinates of the maximum and minimum points on the curve $y = g(x)$.

> Maximum point is $(4, 7)$. ✓
> Minimum point is $(-1, -3)$ ✓

(d) Find the matrix that maps $y = g(x)$ back to $y = f(x)$.

> $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a reflection about the x-axis.
> Hence, reverse transformation has matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. ✓✓

Calculator Free

3. [6 marks: 1, 1, 2, 2] [TISC]

The transformation T is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(a) Describe in words the transformation T.

> A reflection about the x-axis. ✓

(b) The transformation T is applied to the line with equation $y = x$. Find the equation of the resulting line.

> $y = -x$ ✓

(c) The point A is mapped to the point with coordinates $(k, k + 1)$ under transformation T. Find the coordinates of the point A. Justify your answer.

> Since T is a reflection about the x-axis, ✓
> the coordinates of A is $(k, -k - 1)$. ✓

(d) The transformation T is combined with the transformation represented by matrix **M**. All the entries in matrix **M** are positive. The effects of the combined transformation is represented by the matrix $\begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$.

Find the matrix **M**. Show clearly your reasoning.

> $\mathbf{M}\times\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}. \Rightarrow \mathbf{M} = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1}$
> $= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$
> But all entries in **M** are positive. Hence $\mathbf{M} \neq \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$. ✓
>
> $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\times\mathbf{M} = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}. \Rightarrow \mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$
> $= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$ ✓

Calculator Assumed

4. [7 marks: 3, 2, 1, 1] [TISC]

Triangle ABC with vertices A (0, 0), B (3, 0) and C (3, 3) is mapped to triangle A′B′C′ by the compound transformation represented by the matrix

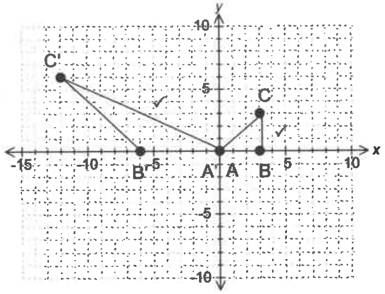
$\mathbf{M} = \begin{pmatrix} -2 & -2 \\ 0 & 2 \end{pmatrix}$.

(a) Find the coordinates of the points A′, B′ and C′.

$$\begin{pmatrix} -2 & -2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 3 & 3 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -6 & -12 \\ 0 & 0 & 6 \end{pmatrix}$$

Hence, A′(0, 0) , B′(−6, 0) and C′(−12, 6) ✓✓✓

(b) Plot triangle ABC and triangle A′B′C′ on the axes provided below.



(c) The transformation applied is a combination of several simple transformations. What evidence is there to suggest that one of the simple transformations involved is :

(i) an enlargement?

The size of the image is larger than the size of the object ✓

(ii) a reflection ?

The vertices of the object is read in an anticlockwise manner while the vertices of the image is read in a clockwise manner. ✓
Or
The *x*-coordinates of the each vertex have their signs reversed whereas the signs of the *y*-coordinates remain unchanged

Calculator Assumed

5. [6 marks]

Let the matrices $\mathbf{R}_\theta$ and $\mathbf{R}_\phi$ represent the linear transformations of an anticlockwise rotation of θ radians and ϕ radians respectively about the origin. Use these two transformations to prove that:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

and $$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi\,.$$

$$\mathbf{R}_\theta\mathbf{R}_\phi = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix}$$ ✓✓

But $\mathbf{R}_\theta\mathbf{R}_\phi = \mathbf{R}_{\theta+\phi}$
$$= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}$$ ✓

Hence, $\begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix}$ ✓

Therefore:
$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ ✓

and $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$ ✓

Calculator Assumed

6. [9 marks: 3, 2, 4] [TISC]

Consider two matrices $\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{T} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

The point A(1, k) is mapped to the point A′′ using **T** followed by **S** as transformation matrices.

(a) Find the coordinates of A′′.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 2 \\ -2k \end{pmatrix}$$ ✓✓

Hence, coordinates of A′′ are (2, −2k). ✓

(b) Find a single transformation matrix that will map A′′ back to A. Show how you obtained your answer.

Hence, reverse mapping has matrix $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ✓ ✓

(c) C is a circle of radius 1 with centre at (1, 1). C is transformed to circle C′ by the transformation **T**. Discuss the differences between the original circle C and its image C′. You need to comment on the coordinates of the centre, the radius and area of the two circles.

Centre is now at (2, 2). ✓

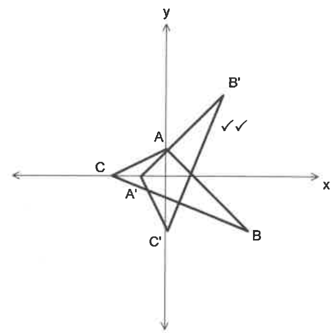
Radius is now 2 × 1 = 2 units ✓

Area is now *four* times larger ie 4π square units. ✓✓

Calculator Assumed

7. [8 marks: 2, 2, 2, 2]

The transformations **P** and **Q** represented by the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ respectively, are applied on ΔABC shown in the diagram below



(a) On the same axes, draw the image of ΔABC under the transformation **P**.

(b) Explain clearly why applying these two transformations in a different order will result in different images.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$$ ✓

Hence **PQ** ≠ **QP** and will result in different images. ✓

(c) Given that the area of ΔABC is k units^2, find the area of the image of ΔABC under the transformations **Q** followed by **P**.

$|\mathbf{PQ}| = 2$ ✓

Hence, area = $2k$ unit^2. ✓

(d) State the matrix of one possible linear transformation **R**, such that the order of application **PR** and **RP** gives identical results.

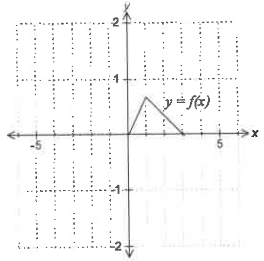
$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is the matrix for **R** an enlargement of factor 2. ✓

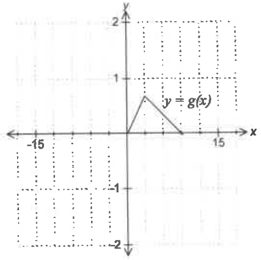
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$ ✓

Calculator Assumed

8. [5 marks: 2, 3]

(a) The graphs of $y = f(x)$ and $y = g(x)$ are drawn below.



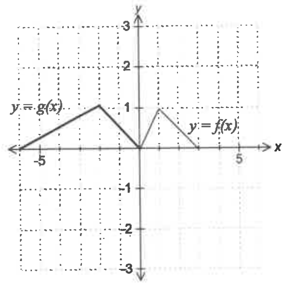


State the matrix representing the linear transformation that maps $y = f(x)$ to $y = g(x)$.

Transformation is a dilation along the x-axis of factor 3.

Hence $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. ✓✓

(b) The graphs of $y = f(x)$ and $y = g(x)$ are drawn below.



State the matrices representing the linear transformations that maps $y = f(x)$ to $y = g(x)$. Discuss the order in which these transformations are applied.

Transformations are: Reflection about the y-axis and Dilation along the x-axis of factor 2, applied in any order.

Hence, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓ and $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ ✓ applied in any order. ✓

Calculator Assumed

9. [5 marks: 2, 3]

(a) Find the matrix representation of a combination of linear transformations that map the points (1, 0) to (–2, 0) and (2, 1) to (–3, 1).

$\mathbf{M}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 0 & 1 \end{pmatrix}$ ✓

$\mathbf{M} = \begin{pmatrix} -2 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix}$ ✓

(b) Show that it is not possible to find a combination of linear transformations that will map the points (1, 0) to (3, 0), (2, 1) to (4, 1) and (3, 1) to (5, 1).

$\mathbf{M}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}$

$\mathbf{M} = \begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$ (I) ✓

Using **M** from (I):

$\mathbf{M}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ ✓

$\begin{pmatrix} 7 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ ✓

Hence, not possible to find a single matrix that will perform the required transformations.

OR

$\mathbf{M}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}$

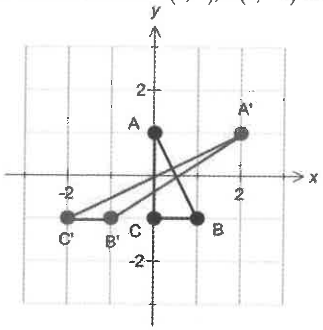
$\mathbf{M} = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -4 \\ 0 & 1 \end{pmatrix}$ (II) ✓

(I) and (II) show that two different matrices are required. ✓
Hence, not possible to find a single matrix that will perform the required transformations.

Calculator Assumed

10. [10 marks: 4, 6]

The triangle ABC has vertices A(0, 1), B(1, −1) and C(0, −1).



(a) ΔABC is mapped to ΔA′B′C′ by a transformation represented by the matrix $\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$. Draw and clearly label ΔA′B′C′ on the diagram above.

> $\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\1 & -1 & -1\end{pmatrix}=\begin{pmatrix}2 & -1 & -2\\1 & -1 & -1\end{pmatrix}$ ✓
>
> Hence A′ (2, 1), B′ (−1, −1) and C′ (−2, −1). ✓✓✓ Marked on diagram.

(b) ΔA′B′C′ is mapped to ΔA″B″C″ by a transformation represented by the matrix $\mathbf{M}=\begin{pmatrix}a & b\\c & d\end{pmatrix}$. The points A″, B″ and C″ have coordinates (2, 3), (−1, k) and (k, −3) respectively. Find matrix **M** and the value(s) of k.

> $\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}2 & -1 & -2\\1 & -1 & -1\end{pmatrix}=\begin{pmatrix}2 & -1 & k\\3 & k & -3\end{pmatrix}$
>
> $2a+b=2$ (1)
> $-a-b=-1$ (2) ✓ Method
> $\Rightarrow a=1, b=0$ ✓
>
> But $-2a-b=k \Rightarrow k=-2$ ✓
>
> $2c+d=3$ (3)
> $-c-d=k=-2$ (4) ✓ Method
> $\Rightarrow c=1, d=1$ ✓
>
> Check: $-2c-d=-3 \Rightarrow -2(1)-1=-3$ True. ✓
>
> Hence, $\mathbf{M}=\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}$ and $k=-2$.

29 Complex Numbers

Calculator Free

1. [7 marks: 1, 2, 2, 2]

Solve for x.

(a) $x^2+4=0$

> $x^2=-4 \Rightarrow x=\pm 2i$ ✓

(b) $x^2+x+4=0$

> $x=\dfrac{-1\pm\sqrt{1^2-4\times1\times4}}{2}$ ✓
>
> $=\dfrac{-1\pm i\sqrt{15}}{2}$ ✓

(c) $(x+2)^2+9=0$

> $(x+2)=\pm 3i$ ✓
> $x=-2\pm 3i$ ✓

(d) $x^3+9x=0$

> $x(x^2+9)=0$
> $x=0, \pm 3i$ ✓✓

2. [5 marks: 1, 4]

Factorise using both real and complex factors where appropriate:

(a) x^2+16

> $x^2+16\equiv(x-4i)(x+4i)$ ✓

(b) x^2+2x+5

> Zeroes are: $x=\dfrac{-2\pm\sqrt{2^2-4\times1\times5}}{2}$ ✓
>
> $=-1\pm 2i$ ✓
>
> Hence: $x^2+2x-1\equiv[x-(-1+2i)][x-(-1-2i)]$
> $\equiv(x+1-2i)(x+1+2i)$ ✓✓

Calculator Free

3. [5 marks: 2, 3]

Express in the form $a + bi$:

(a) $(1 + i\sqrt{2})^2$

$(1 + i\sqrt{2})^2 = 1 + 2\sqrt{2}\,i + (i\sqrt{2})^2$ ✓✓
$= -1 + 2\sqrt{2}\,i$

(b) $\dfrac{2+3i}{1-i}$

$\dfrac{2+3i}{1-i} = \dfrac{(2+3i)(1+i)}{(1-i)(1+i)}$ ✓
$= \dfrac{-1}{2} + \dfrac{5i}{2}$ ✓✓

---

4. [5 marks]

Given that $(a + b\,i)^2 = -15 + 8\,i$, find a and b where a and b are real non-zero integers.

$a^2 - b^2 + 2abi = -15 + 8i$ ✓

Hence: $a^2 - b^2 = -15$ ✓
$ab = 4$ ✓

By inspection: $a = 1,\ b = 4$ ✓
$a = -1,\ b = -4$ ✓

Calculator Free

5. [7 marks: 1, 3, 3]

Let $u = 1 + 3\,i$, $v = -1 + i$ and $w = -2\,i$.

(a) Find $u + v$.

$u + v = 4\,i$ ✓

(b) Find $\dfrac{v}{w}$.

$\dfrac{v}{w} = \dfrac{-1+i}{-2i}$
$= \dfrac{-1+i}{-2i} \times \dfrac{i}{i}$ ✓
$= \dfrac{-1-i}{2}$ ✓✓

(c) Find $v \times \bar{u}$.

$(-1 + i) \times (1 - 3\,i)$ ✓
$= -1 + 3\,i + i + 3$
$= 2 + 4\,i$ ✓✓

---

6. [8 marks: 4, 4]

Find $w = a + b\,i$ where a and b are real non-zero integers if:

(a) $\bar{w} = \dfrac{5}{w}$,

$w = a + b\,i \Rightarrow \bar{w} = a - b\,i$ ✓
Hence $(a + b\,i) \times (a - b\,i) = 5$
$\Rightarrow a^2 + b^2 = 5$ ✓
Since, a and b are real non-zero integers, $a = \pm 2, b = \pm 1$ ✓
or $b = \pm 2, a = \pm 1$ ✓

(b) $w^2 = -3 - 4\,i$

$(a + b\,i)^2 \equiv a^2 - b^2 + 2ab\,i = -3 - 4\,i$
Compare Re parts: $a^2 - b^2 = -3$ ✓
Compare Im parts: $ab = -2$ ✓
Since, a and b are real non-zero integers, $a = 1, b = -2$ ✓
$a = -1, b = 2$ ✓

Calculator Free

7. [9 marks: 3, 3, 3]

Let the complex numbers $z_1 = 2 + k\,i$ and $z_2 = -5 + 12\,i$, where k is a *real number*. Determine all possible values of k if:

(a) $[\,\boldsymbol{Im}(z_1)\,]^2 = \boldsymbol{Im}(z_2)$

$k^2 = 12$ ✓
$\Rightarrow\ k = \pm 2\sqrt{3}$ ✓✓

(b) $z_1^{\,2} = z_2$

$(2 + k\,i)^2 = -5 + 12\,i$
$(4 - k^2) + 4\,k\,i = -5 + 12\,i$ ✓
Equate Real parts: $4 - k^2 = -5 \Rightarrow k = \pm 3$ ✓
Equate Im parts $4k = 12 \Rightarrow k = 3$
Hence, $k = 3$. ✓

(c) $\dfrac{6z_1}{z_2} = -i$.

$6\,z_1 = -i \times z_2$
$6\,(2 + k\,i) = -i \times (-5 + 12\,i)$ ✓
$12 + 6\,k\,i = 12 + 5\,i$ ✓
$\Rightarrow\ k = 5/6$ ✓

8. [8 marks: 2, 6]

Let the complex numbers $u = 3 + k\,i$ and $v = k + 2\,i$, where k is a *real number*. Determine all possible values of k if:

(a) $[\,\boldsymbol{Im}(u)\,]^2 = \boldsymbol{Re}(v)$

$k^2 = k$
$k = 0, 1$ ✓✓

(b) $\dfrac{u}{v-1} = -1 - \dfrac{i}{2}$

$(3 + ki) = (k + 2i - 1)\left(-1 - \dfrac{i}{2}\right)$

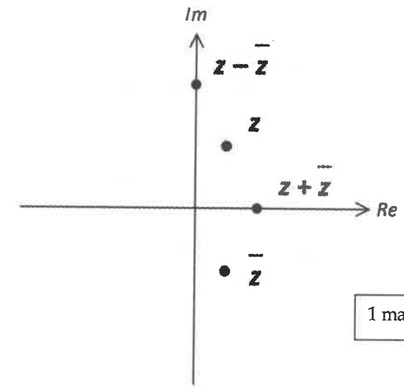
$3 + ki = (k - 1 + 2i)\left(-1 - \dfrac{i}{2}\right)$ ✓

Real Parts: $3 = -(k-1) + 1 \Rightarrow k = -1$ ✓✓
Im Parts: $k = -2 - \dfrac{k-1}{2} \Rightarrow k = -1$ ✓✓
Hence, $k = -1$. ✓

Calculator Free

9. [3 marks]

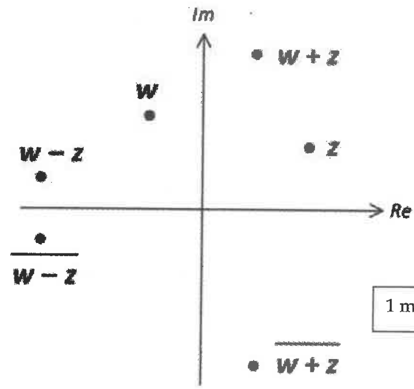
The complex number z is plotted in the Argand plane shown below. In the Argand plane provided below, locate and label the points corresponding to: $\overline{z}$, $z + \overline{z}$ and $z - \overline{z}$.



1 mark each ✓✓✓

10. [4 marks]

The complex numbers w and z are plotted in the Argand plane shown below. In the Argand plane provided below, locate and label the points corresponding to: $w + z$, $\overline{w+z}$, $w - z$ and $\overline{w-z}$.



1 mark each ✓✓✓✓

30 Conjectures & Proofs

Calculator Free

1. [10 marks: 2, 2, 3, 3]

Provide a counter-example to disprove each of the following conjectures.

(a) If $x^2 > 100$, then $x > 10$.

> For $x = -12$: $x^2 > 100$ but $x \ngtr 10$. ✓✓

(b) If $2 > x$, then $2^2 > x^2$.

> For $x = -3$: $2 > -3$ but $2^2 \ngtr (-3)^2$. ✓✓

(c) The sum of two numbers is always greater than the larger of the two numbers.

> Let the two numbers be 10 and −10. ✓✓
> The larger of these two numbers is 10.
> The sum of these two numbers is 0.
> But $0 \ngtr 10$. ✓

(d) The sum of any two positive numbers must always be less than the product of these two numbers.

> Let the two numbers be 10 and 0.1. ✓✓
> The sum of these two numbers is 10.1.
> The product of these two numbers is 1.
> But $10.1 \nless 1$.. ✓

Calculator Free

2. [5 marks: 1, 1, 3]

(a) Given that $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$. Find:

(i) $\mathbf{P} \times \mathbf{Q}$

> $\mathbf{P} \times \mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

(ii) $\mathbf{Q} \times \mathbf{P}$

> $\mathbf{Q} \times \mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

(b) Prove or disprove the conjecture that if the matrices **P** and **Q** are such that $\mathbf{P} \times \mathbf{Q} = \mathbf{I}$, where **I** is the relevant identity matrix, then $\mathbf{P}^{-1} = \mathbf{Q}$.

> Conjecture is false. ✓
>
> Counter-example:
>
> From (a) (i):
>
> $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓
>
> But $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ does not have an inverse as it is not a square matrix. ✓

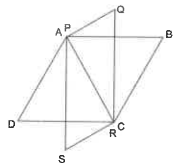
Calculator Free

3. [9 marks: 3, 3, 3]

Provide a counter-example to disprove each of the following conjectures.

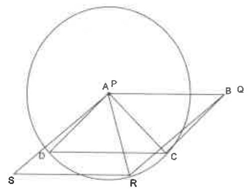
(a) Consider the parallelograms ABCD and PQRS. If the diagonals AC = PR then ABCD and PQRS are congruent.

As shown in the accompanying diagram, AC = PR but ABCD and PQRS are not congruent. ✓✓✓



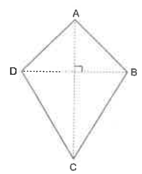
(b) Consider the parallelograms ABCD and PQRS. If AB = PQ and the diagonals AC = PR, ABCD and PQRS are congruent.

As shown in the accompanying diagram, AB = PQ and diagonals AC = PR, but ABCD and PQRS are not congruent. ✓✓✓



(c) If the diagonals of a quadrilateral are perpendicular then the quadrilateral must be a rhombus.

As shown in the accompanying diagram, the diagonals AC and BD are perpendicular but ABCD is not a rhombus. ✓✓✓



Calculator Assumed

4. [12 marks: 2, 2, 4, 4]

Provide a counter-example to disprove each of the following conjectures. Show all attempts, successful and otherwise.

(a) $2^{2n} + 1$ is prime for integer $n \geq 1$.

$n = 3$, $2^{2n} + 1 = 65$
$= 13 \times 5$ which is not prime. ✓✓

(b) $n^2 - n + 5$ is prime for integer $n \geq 1$.

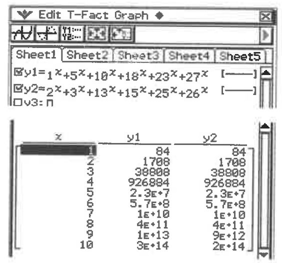
$n = 5$, $n^2 - n + 5 = 25$
$= 5 \times 5$ which is not prime. ✓✓

(c) $1^n + 5^n + 10^n + 18^n + 23^n + 27^n = 2^n + 3^n + 13^n + 15^n + 25^n + 26^n$ for integer $n \geq 1$.

$n = 9$. ✓✓

LHS $\approx 1 \times 10^{13}$ ✓
RHS $\approx 9 \times 10^{12}$ ✓

Hence, LHS ≠ RHS



(d) There are no integer solutions to $a^3 + b^3 = c^3 + d^3$ where $a \neq b \neq c \neq d$.

For $a = 1$, $b = -1$, $c = 2$, $d = -2$: ✓✓✓✓
LHS = 0
RHS = 0.

Calculator Free

5. [3 marks]

Prove that $2^n - 1$ is always odd for integer $n \geq 1$.

2^n for $n \geq 1$ will always be a multiple of 2	
and hence is even for $n \geq 1$.	✓✓
Hence, $2^n - 1$ will always be odd.	✓

6. [3 marks]

Prove that the square of an odd number add 11 is a multiple of 4.

$(2k+1)^2 + 11 = 4k^2 + 4k + 12$	✓✓
$= 4(k^2 + k + 3)$	✓
which is a multiple of 4.	

7. [5 marks: 2, 3]

(a) Prove that product of three consecutive integers is divisible by 3.

In a run of three consecutive integers, one	
of the integers must be a multiple of 3.	✓
Hence, the product must be a multiple of 3.	✓

(b) Prove that product of three consecutive integers is divisible by 6.

In a run of three consecutive integers, one	
of the integers must be a multiple of 3	✓
and at least one of the integers must be even.	✓
Hence, the product must be a multiple of 3 & 2.	
That is, the product must be multiple of 6	✓

Calculator Assumed

8. [3 marks]

Prove that $x^7 - x$ is divisible by 6 for integer $x \geq 1$.

$x^7 - x = (x-1)\,x\,(x+1)\,(x^2+x+1)(x^2-x+1)$	✓
But $(x-1)\,x\,(x+1)$ is a product of 3 consecutive integers.	
As one of the three integers must be a multiple of 3,	
and at least one must be even,	
the product must be a multiple of 6.	✓
Hence, $(x-1)\,x\,(x+1) = 6k$.	
$\Rightarrow x^7 - x = 6k\,(x^2+x+1)(x^2-x+1)$	✓
= multiple of 6.	

9. [4 marks]

Prove that the product of any three consecutive even numbers must be a multiple of 24.

$(2n) \times (2n+2) \times (2n+4) = 8 \times n(n+1)(n+2)$	✓
But $n(n+1)(n+2)$ = product of 3 consecutive integers	
= multiple of 3 as one of the three integers must be a multiple of 3	
$= 3k$	✓
Hence:	
$(2n) \times (2n+2) \times (2n+4) = 8 \times n(n+1)(n+2)$	
$= 8 \times 3k$	✓
$= 24k$	✓
= multiple of 24.	

10. [3 marks]

Prove that $4n^3 - 4n$ is a multiple of 24 for integer $n \geq 1$.

$4n^3 - 4n = 4n(n^2 - 1)$	
$= 4n(n-1)(n+1)$	✓
$(n-1)n(n+1)$ is the product of 3 consecutive integers, one of which must be a multiple of 3 and one of which must be even.	
Hence, $(n-1)n(n+1)$ must be a multiple of 6.	✓
$\Rightarrow 4n(n-1)(n+1)$ = multiple of 4×6,	✓
Hence, $4n^3 - 4n$ is a multiple of 24.	

Calculator Assumed

11. [7 marks: 4, 3]

(a) It is conjectured that a number is divisible by 4 if the sum of twice the tens digit and the ones digit is a multiple of 4. Prove that this conjecture is true for a four digit number.

Let the number be *abcd*.
Value of number $V = 1000a + 100b + 10c + d$ ✓

Required sum in conjecture $= 2c + d$. ✓
If $2c + d$ is a multiple of 4: $2c + d = 4k$
$d = 4k - 2c$ ✓

Hence, $V = 1000a + 100b + 10c + 4k - 2c$
$= 1000a + 100b + 8c + 4k$
$= 4(250a + 25b + 2c + 1)$ ✓
$\Rightarrow$ V is a multiple of 4, and *abcd* is a multiple of 4.

(b) Consider the positive integers a and b. The arithmetic mean of these two integers is $M = \dfrac{a+b}{2}$ and the harmonic mean is $H = \dfrac{2ab}{a+b}$.

It is conjectured that $M \geq H$. Use the expansion of $(\sqrt{a} - \sqrt{b})^2$ to prove this conjecture.

$(\sqrt{a} - \sqrt{b})^2 = a + b - 2\sqrt{ab}$
Clearly $(\sqrt{a} - \sqrt{b})^2 \geq 0$ ✓
Hence, $a + b - 2\sqrt{ab} \geq 0$
$a + b \geq 2\sqrt{ab}$ ✓
$(a + b)^2 \geq 4ab$
Hence: $\dfrac{a+b}{2} \geq \dfrac{2ab}{a+b}$ ✓
That is $M \geq H$.

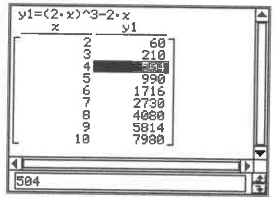
Calculator Assumed

12. [7 marks: 2, 5]

(a) Provide a counter-example to disprove the conjecture that the cube of an even number greater than two less the number itself is divisible by 5.

Number $= (2n)^3 - 2n$ for $n \geq 2$

Counter example: $n = 4$ Number $= 8^3 - 8 = 504$ ✓✓
which is not divisible by 5.



(b) Prove that $x^5 + x^4 + x^3 + x^2$ is divisible by 4 for x as a whole number.

Expression $= x^5 + x^4 + x^3 + x^2 = x^2(x^2 + 1)(x + 1)$ ✓

If x is an even number: $x = 2n$ ✓
Expression $= (2n)^2\,[(2n)^2 + 1](2n + 1)$
$= 4n^2(4n^2 + 1)(2n + 1)$ ✓
which is divisible by 4.

If x is an odd number: $x = 2n + 1$ ✓
Expression $= (2n + 1)^2\,[(2n + 1)^2 + 1](2n + 1 + 1)$
$= (2n + 1)^2\,[4n^2 + 4n + 1 + 1](2n + 1 + 1)$
$= (2n + 1)^2\,[4n^2 + 4n + 2](2n + 2)$
$= 4\,(2n + 1)^2\,(2n^2 + 2n + 1)(n + 1)$ ✓
which is divisible by 4.

Hence, expression is always divisible by 4.

Calculator Assumed

13. [7 marks: 1, 1, 5] [TISC]

Consider the conjecture that the product of any two positive numbers must be greater than the sum of these two numbers.

(a) Provide an example that proves this conjecture.

Let the numbers be 10 and 20.
Then Product = 10 × 20 = 200.
Sum = 10 + 20 = 30.
Clearly, Product > Sum. ✓

(b) Provide an example that disproves this conjecture.

Let the numbers be 1 and 2.
Then Product = 1 × 2 = 2.
Sum = 1 + 2 = 3.
Clearly, Product ≯ Sum. ✓

(c) Under what conditions will this conjecture be always true?

Let the numbers be x and y.
$\Rightarrow \quad xy > x + y$ ✓
$xy - x > y$
$x(y-1) > y$ ✓
If $y > 1$, then there will be no sign change and $x > \dfrac{y}{y-1}$. ✓

Hence, conjecture will always be true if $x > \dfrac{y}{y-1}$ where $y > 1$. ✓✓

Calculator Assumed

14. [7 marks: 2, 2, 3] [TISC]

Consider the conjecture that the *quotient* of any two positive numbers must be less than the *product* of these two numbers. That is: if one positive number is divided by another positive number, then the result must be less than the *product* of these two numbers.

(a) Provide an example that supports this conjecture.

Let the numbers be 10 and 20.
Then Product = 10 × 20 = 200.
Quotient = 10 ÷ 20 = 0.5
Quotient = 20 ÷ 10 = 2. ✓ Must show both Quotients
Clearly, Product > Quotient. ✓

(b) Provide an example that disproves this conjecture.

Let the numbers be $\frac{1}{2}$ and $\frac{1}{5}$.

Then Product = $\frac{1}{2} \times \frac{1}{5} = \frac{1}{10}$.

Quotient = $\frac{1}{2} \div \frac{1}{5} = \frac{5}{2}$.

Quotient = $\frac{1}{5} \div \frac{1}{2} = \frac{2}{5}$ ✓ Must show both Quotients

Clearly, Product ≯ Quotient. ✓

(c) Under what conditions will this conjecture be always true? Justify your answer.

Let the numbers be x and y.

$\Rightarrow \quad xy > \frac{x}{y}$ and $xy > \frac{y}{x}$ ✓

For $xy > \frac{x}{y}$

As $y > 0$ $\quad xy^2 - x > 0$
$x(y^2 - 1) > 0$
As $x > 0$ $\quad y > 1$ ✓

Similarly $xy > \frac{y}{x} \Rightarrow x > 1$

Hence, conjecture is always true if $x > 1$ and $y > 1$. ✓

Calculator Assumed

15. [10 marks: 3, 3, 4]

(a) Prove that $15.\overline{15}$ is a rational number.

Let	$n = 15.\overline{15}$	I	✓
	$100n = 1515.\overline{15}$	II	✓
II – I	$99n = 1500$		
Hence:	$n = \frac{1500}{99} = \frac{500}{33}$.		✓

Therefore, $15.\overline{15}$ is a rational number.

(b) Prove that $5.7\overline{35}$ is a rational number.

Let	$n = 5.7\overline{35}$	I	
	$10n = 57.\overline{35}$	II	✓
	$1000n = 5735.\overline{35}$	III	✓
III – II	$990n = 5678$		
Hence:	$n = \frac{5678}{990} = \frac{2839}{495}$		✓

Therefore, $5.7\overline{35}$ is a rational number.

(c) Prove that $10 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \ldots$ is a rational number.

The terms of the series,

$10, \frac{1}{10}, \frac{1}{10^2}, \frac{1}{10^3}, \frac{1}{10^4}, \ldots\ldots$

form an infinite geometric sequence

with first term $a = 10$ and common ratio $r = \frac{1}{10}$. ✓✓

Since $|r| < 1$, the infinite sequence has a finite sum. ✓

Sum of infinite sequence $= \frac{a}{1-r}$

$= \frac{10}{1 - \frac{1}{10}}$

$= \frac{100}{9}$. ✓

Hence, series sum is rational.

Calculator Assumed

16. [9 marks: 5, 4]

(a) Negate the converse of the following conjecture:

If n is a multiple of 10, then n^2 is a multiple of 10.

Prove that the converse is true by proving that the negation of the converse is false.

Converse of conjecture:

If n^2 is a multiple of 10, then n is a multiple of 10. ✓

Assume that the negation of the converse is true:

Given that n^2 is a multiple of 10, then n is <u>not</u> a multiple of 10. ✓

As n is not a multiple of 10, $n = 10a + b$

for some integer a and integer b where $1 \le b \le 9$. ✓

Hence, $n^2 = (10a + b)^2$

$= 100a^2 + 20ab + b^2$

$= 10(10a^2 + 2ab) + b^2$ I ✓

As n^2 is a multiple of 10, b^2 in statement I must also be a multiple of 10.

Which is a contradiction as for $1 \le b \le 9$, b^2 is not a multiple of 10. ✓

Therefore the negation of the converse cannot be true.

Hence, if n^2 is a multiple of 10, then n must be a multiple of 10.

(b) Use the result from (a) and the method of contradiction to prove that $\sqrt{10}$ is an irrational number.

Assume that $\sqrt{10}$ is rational.

$\Rightarrow \sqrt{10} = \frac{a}{b}$, where a and b are integers with no factors in common. ✓

$\Rightarrow a^2 = 10b^2$

$\Rightarrow a^2$ and hence a are multiples of 10 (from (a)). ✓

Hence, $a = 10k$ for some integer k.

$\Rightarrow (10k)^2 = 10b^2$

$\Rightarrow b^2 = 10k^2$

$\Rightarrow b^2$ and hence b are multiples of 10 (from (a)). ✓

Hence, a and b are both multiples of 10;

which contradicts the initial premise that a and b have no factors in common. ✓

Hence, $\sqrt{10}$ must be irrational.

Calculator Assumed

17. [10 marks: 5, 5]

(a) Consider the conjecture:

If m is a rational number and n is an irrational number, then $m + n$ is an irrational number.

State the negation of this conjecture.

Prove that this conjecture is true by proving that its negation is false.

Negation of conjecture:
If m is a rational number and n is an irrational number, then $m + n$ is rational. ✓

If $m + n$ is rational,

$\Rightarrow m + n = \frac{a}{b}$, where a and b are integers with no factors in common. ✓

Hence, $n = \frac{a}{b} - m$.

Since m is rational, $m = \frac{c}{d}$ ✓

Hence, $n = \frac{a}{b} - \frac{c}{d}$ ✓

This makes n rational which contradicts the statement that n is irrational. ✓

Hence, the negation is false and thence $m + n$ must be irrational.

(b) Prove that if m is a rational number and n is an irrational number, then $m \times n$ is an irrational number.

Assume that the negation is true:
If m is a rational number and n is an irrational number, then $m \times n$ is rational. ✓

If $m \times n$ is rational,

$\Rightarrow m \times n = \frac{a}{b}$, where a and b are integers with no factors in common. ✓

Hence, $n = \frac{a}{b} \times \frac{1}{m}$

Since m is rational, $m = \frac{c}{d}$ ✓

Hence, $n = \frac{a}{b} \times \frac{d}{c}$ ✓

This makes n rational which contradicts the statement that n is irrational. ✓

Hence, the negation is false and $m + n$ must be irrational.

Calculator Assumed

18. [7 marks: 5, 2]

(a) Use mathematical induction to prove that:

$$\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \ldots + \frac{n}{5^n} = \frac{5}{16} - \frac{4n+5}{16(5^n)} \text{ for integer } n \geq 1.$$

For $n = 1$: $\quad \text{LHS} = \frac{1}{5}$, $\text{RHS} = \frac{5}{16} - \frac{9}{80} = \frac{1}{5}$. ✓

Hence, conjecture is true for $n = 1$.

Assume that the conjecture is true for $n = k$.

That is: $\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \ldots + \frac{k}{5^k} = \frac{5}{16} - \frac{4k+5}{16(5^k)}$.

For $n = k + 1$: $\quad \text{LHS} = [\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \ldots + \frac{k}{5^k}] + \frac{k+1}{5^{k+1}}$

$= \frac{5}{16} - \frac{4k+5}{16(5^k)} + \frac{k+1}{5^{k+1}}$

$= \frac{5}{16} + \frac{-5(4k+5)+16(k+1)}{16(5^{k+1})}$ ✓

$= \frac{5}{16} + \frac{-4k-9}{16(5^{k+1})}$

$= \frac{5}{16} - \frac{4(k+1)+5}{16(5^{k+1})}$. ✓✓

$\text{RHS} = \frac{5}{16} - \frac{4(k+1)+5}{16(5^{k+1})}$.

Hence, if it is assumed true for $n = k$, it will be true for $n = k + 1$. ✓

Since, it is true for $n = 1$, then it must be true for $n = 2$ and subsequent integers.

Hence, conjecture is true for all integers $n \geq 1$.

(b) Discuss the sum as the number of terms increases indefinitely.

As $n \to \infty$, sum $= \frac{5}{16} - \frac{4n+5}{16(5^n)} \to \frac{5}{16}$. ✓✓

Calculator Assumed

19. [5 marks]

Prove inductively that $(1+i)^{4n} = (-1)^n\, 2^{2n}$ for integer $n \ge 1$.

For $n = 1$: LHS $= (1+i)^4 = -4$, RHS $= (-1)^1 2^2 = -4$. ✓
Hence, conjecture is true for $n = 1$.

Assume that the conjecture is true for $n = k$.
That is: $(1+i)^{4k} = (-1)^k\, 2^{2k}$
For $n = k+1$: LHS $= (1+i)^{4(k+1)}$
$= (1+i)^{4k} \times (1+i)^4$ ✓
$= (-1)^k\, 2^{2k} \times -4$ ✓
$= (-1)^{k+1}\, 2^{2(k+1)}$ ✓
RHS $= (-1)^{k+1}\, 2^{2(k+1)}$

Hence, if it is assumed true for $n = k$, it will be true for $n = k+1$. ✓
Since, it is true for $n = 1$, then it must be true for $n = 2$ and subsequent integers.

Hence, conjecture is true for all integers $n \ge 1$.

20. [5 marks]

Use mathematical induction to prove that $11^n + 4$ is divisible by 5 for integer $n \ge 1$.

For $n = 1$: $11^n + 4 = 15$ which is divisible by 5 ✓
Hence, conjecture is true for $n = 1$.

Assume that the conjecture is true for $n = k$.
That is: $11^k + 4 = 5m$ for integer $m \ge 1$.
For $n = k+1$: Expression $= 11^{k+1} + 4$
$= 11(11^k) + 4$ ✓
$= 11(5m - 4) + 4$ ✓
$= 55m - 40$
$= 5(11m - 8)$ which is divisible by 5. ✓

Hence, if it is assumed true for $n = k$, it will be true for $n = k+1$. ✓
Since, it is true for $n = 1$, then it must be true for $n = 2$ and subsequent integers.

Hence, conjecture is true for all integers $n \ge 1$.

Calculator Assumed

21. [6 marks]

Prove that $10^{n+1} + 3 \times 10^n + 5$ is a multiple of 9 for positive integer n.

For $n = 1$, Expression $= 10^2 + 3 \times 10 + 5$
$= 135 = 9 \times 15$
Hence, conjecture is true for $n = 1$. ✓

Assume that the conjecture is true for $n = k$.
That is, $10^{k+1} + 3 \times 10^k + 5 = 9x$ for some integer x **I** ✓

For $n = k+1$:
Expression $= 10^{k+2} + 3 \times 10^{k+1} + 5$ ✓
$= 10(10^{k+1}) + 30(10^k) + 5$

But from I, $10^{k+1} = 9x - 5 - 3(10^k)$ ✓

Hence, Expression $= 10[9x - 5 - 3(10^k)] + 30(10^k) + 5$
$= 90x - 45 = 9(10x - 5)$ which is a multiple of 9. ✓

Hence, if it is assumed true for $n = x$, it will be true for $n = x + 1$. ✓
Since, it is true for $n = 1$, then it must be true for $n = 2$ and subsequent integers.

Hence, conjecture is true for all integers $n \ge 1$.

22. [4 marks]

Given the non-singular commutative matrices **P** and **Q**, use mathematical induction to prove that for integer $n \ge 1$, $\mathbf{P}^n = \mathbf{Q}\,\mathbf{P}^n\,\mathbf{Q}^{-1}$.

For $n = 1$: LHS $= \mathbf{P}$ RHS $= \mathbf{Q\,P\,Q}^{-1} = \mathbf{P\,Q\,Q}^{-1} = \mathbf{P}$. ✓
Hence, conjecture is true for $n = 1$.

Assume that the conjecture is true for $n = k$.
That is: $\mathbf{P}^k = \mathbf{Q}\,\mathbf{P}^k\,\mathbf{Q}^{-1}$.
For $n = k+1$: RHS $= \mathbf{Q}\,\mathbf{P}^{k+1}\,\mathbf{Q}^{-1}$
$= \mathbf{Q\,P\,P}^k\,\mathbf{Q}^{-1}$ ✓
$= \mathbf{P}\,(\mathbf{Q}\,\mathbf{P}^k\,\mathbf{Q}^{-1})$ ✓
$= \mathbf{P\,P}^k = \mathbf{P}^{k+1}$
LHS $= \mathbf{P}^{k+1}$

Hence, if it is assumed true for $n = k$, it will be true for $n = k+1$. ✓
Since, it is true for $n = 1$, then it must be true for $n = 2$ and subsequent integers.
Hence, conjecture is true for all integers $n \ge 1$.

Calculator Assumed

23. [5 marks] [TISC]

Use mathematical induction to prove that for integer $n \geq 1$:

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2n-1)x = \frac{\sin 2nx}{2\sin x}.$$

[Hint: Use the formula $2\cos A \sin B = \sin(A+B) - \sin(A-B)$.]

For $n = 1$: LHS $= \cos x$

RHS $= \dfrac{\sin 2x}{2\sin x}$

$= \dfrac{2\sin x\cos x}{2\sin x}$

$= \cos x$. ✓

Hence, conjecture is true for $n = 1$.

Assume that the conjecture is true for $n = k$.
That is:

$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x = \dfrac{\sin 2kx}{2\sin x}$. ✓

For $n = k + 1$:

LHS $= \cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x + \cos(2k+1)x$

$= \dfrac{\sin 2kx}{2\sin x} + \cos(2k+1)$

$= \dfrac{\sin 2kx + 2\times[\cos(2k+1)x]\sin x}{2\sin x}$ ✓

$= \dfrac{\sin 2kx + \{[\sin(2kx+x+x)] - [\sin(2kx+x-x)]\}}{2\sin x}$ ✓

$= \dfrac{\sin 2(k+1)x}{2\sin x}$.

RHS $= \dfrac{\sin 2(k+1)x}{2\sin x}$.

Hence, if it is assumed true for $n = k$, it will be true for $n = k + 1$. ✓
Since, it is true for $n = 1$, then it must be true for $n = 2$
and subsequent integers.

Hence, conjecture is true for all integers $n \geq 1$.

Calculator Assumed

24. [4 marks]

Complete the table below. 1 mark each ✓✓✓✓

Conjecture	If x^2 is an odd number then x is an odd number.
Negation of conjecture	If x^2 is an odd number then x is not an odd number.
Contrapositive of conjecture	If x is not an odd number then x^2 is not an odd number.
Converse of conjecture	If x is an odd number then x^2 is an odd number.
Inverse of conjecture	If x^2 is not an odd number then x is not an odd number.

25. [5 marks: 2, 3]

Consider the conjecture:
If a is a factor of 20, then a is also factor of 40.

(a) Prove that the conjecture is true.

a is a factor of 20
$\Rightarrow \exists\ t$ such that $at = 20$ where t is a whole number. ✓

Since $20 \times 2 = 40 \Rightarrow at \times 2 = 40$
$a(2t) = 40$ ✓

Hence, a is also a of 40.

(b) State the inverse of the conjecture and determine with reasons whether it is true or false.

Inverse:
If a is not a factor of 20 $\Rightarrow$ a is not a factor of 40. ✓

Inverse is false. ✓
Counter-example:
40 is not a factor of 20 but is a factor of 40. ✓

Calculator Assumed

26. [7 marks: 5, 2]

Consider the statement: $\tan\theta = 0 \Rightarrow \sin\theta = 0$.

(a) Determine with reasons if the contrapositive of this statement is true or false.

Contrapositive:
$\sin\theta \neq 0 \Rightarrow \tan\theta \neq 0$. ✓

Consider the negation of the contrapositive:
$\sin\theta \neq 0 \Rightarrow \tan\theta = 0$. ✓

Since $\sin\theta \neq 0$, $\sin\theta = k$ $\quad k \neq 0, -1 \leq k \leq 1$, ✓

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{k}{\sqrt{1-k^2}}$$

But $\tan\theta = 0 \Rightarrow \frac{k}{\sqrt{1-k^2}} = 0$

$\Rightarrow k = 0$
which contradicts $k \neq 0$. ✓

Hence, the negation of the contrapositive is false and contrapositive must be true. ✓

(b) Hence or otherwise, determine with reasons if the conjecture is true of false.

Since the contrapositive of the conjecture is true, ✓
the conjecture is true. ✓

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27. [6 marks: 3, 3]

Consider the statement: $\cos\theta = \frac{1}{2} \Rightarrow \tan\theta = \sqrt{3}$

(a) Determine with reasons if the converse of this statement is true or false.

Converse: $\tan\theta = \sqrt{3} \Rightarrow \cos\theta = \frac{1}{2}$. ✓

Converse is false. ✓

Counter-example: $\tan 240^\circ = \sqrt{3}$ but $\cos 240^\circ = -\frac{1}{2} \neq \frac{1}{2}$. ✓

(b) Hence, or otherwise determine with reasons if the inverse is true or false.

Inverse: $\cos\theta \neq \frac{1}{2} \Rightarrow \tan\theta \neq \sqrt{3}$. ✓

Inverse is false as the converse has been shown to be false. ✓✓